ELEMENTS OF TRIGONOMETRY

PLANE AND SPHERICAL

BY

ANDREW W. PHILLIPS, PH.D.

AND

WENDELL M. STRONG, PH.D.

YALE UNIVERSITY

NEW YORK :: CINCINNATI :: CHICAGO

AMERICAN BOOK COMPANY
PREFACE

In this work the trigonometric functions are defined as ratios, but their representation by lines is also introduced at the beginning, because certain parts of the subject can be treated more simply by the line method, or by a combination of the two methods, than by the ratio method alone.

Attention is called to the following features of the book:

The simplicity and directness of the treatment of both the Plane and Spherical Trigonometry.

The emphasis given to the formulas essential to the solution of triangles.

The large number of exercises.

The graphical representation of the trigonometric, inverse trigonometric, and hyperbolic functions.

The use of photo-engravings of models in the Spherical Trigonometry.

The recognition of the rigorous ideas of modern mathematics in dealing with the fundamental series of trigonometry.

The natural treatment of the complex number and the hyperbolic functions.

The graphical solution of spherical triangles.

Our grateful acknowledgments are due to our colleague, Professor James Pierpont, for valuable suggestions regarding the construction of Chapter VI.

We are also indebted to Dr. George T. Sellew for making the collection of miscellaneous exercises.

ANDREW W. PHILLIPS,
WENDELL M. STRONG.

YALE UNIVERSITY. December, 1898.
# Table of Contents

**Plane Trigonometry**

## Chapter I

**The Trigonometric Functions**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angles</td>
<td>1</td>
</tr>
<tr>
<td>Definitions of the Trigonometric Functions</td>
<td>4</td>
</tr>
<tr>
<td>Signs of the Trigonometric Functions</td>
<td>8</td>
</tr>
<tr>
<td>Relations of the Functions</td>
<td>10</td>
</tr>
<tr>
<td>Functions of an Acute Angle of a Right Triangle</td>
<td>13</td>
</tr>
<tr>
<td>Functions of Complementary Angles</td>
<td>14</td>
</tr>
<tr>
<td>Functions of $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$, $360^\circ$</td>
<td>15</td>
</tr>
<tr>
<td>Functions of the Supplement of an Angle</td>
<td>16</td>
</tr>
<tr>
<td>Functions of $45^\circ$, $30^\circ$, $60^\circ$</td>
<td>17</td>
</tr>
<tr>
<td>Functions of $(-x)$, $(180^\circ - x)$, $(180^\circ + x)$, $(360^\circ - x)$</td>
<td>18</td>
</tr>
<tr>
<td>Functions of $(90^\circ - y)$, $(90^\circ + y)$, $(270^\circ - y)$, $(270^\circ + y)$</td>
<td>20</td>
</tr>
</tbody>
</table>

## Chapter II

**The Right Triangle**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution of Right Triangles</td>
<td>22</td>
</tr>
<tr>
<td>Solution of Oblique Triangles by the Aid of Right Triangles</td>
<td>28</td>
</tr>
</tbody>
</table>

## Chapter III

**Trigonometric Analysis**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof of Fundamental Formulas $(11)-(14)$</td>
<td>32</td>
</tr>
<tr>
<td>Tangent of the Sum and Difference of Two Angles</td>
<td>36</td>
</tr>
<tr>
<td>Functions of Twice an Angle</td>
<td>36</td>
</tr>
<tr>
<td>Functions of Half an Angle</td>
<td>36</td>
</tr>
<tr>
<td>Formulas for the Sums and Differences of Functions</td>
<td>37</td>
</tr>
<tr>
<td>The Inverse Trigonometric Functions</td>
<td>39</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS

CHAPTER IV
THE OBLIQUE TRIANGLE

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivation of Formulas</td>
<td>41</td>
</tr>
<tr>
<td>Formulas for the Area of a Triangle</td>
<td>44</td>
</tr>
<tr>
<td>The Ambiguous Case</td>
<td>45</td>
</tr>
<tr>
<td>The Solution of a Triangle:</td>
<td></td>
</tr>
<tr>
<td>(1.) Given a Side and Two Angles</td>
<td>46</td>
</tr>
<tr>
<td>(2.) Given Two Sides and the Angle Opposite One of Them</td>
<td>46</td>
</tr>
<tr>
<td>(3.) Given Two Sides and the Included Angle</td>
<td>48</td>
</tr>
<tr>
<td>(4.) Given the Three Sides</td>
<td>49</td>
</tr>
<tr>
<td>Exercises</td>
<td>50</td>
</tr>
</tbody>
</table>

CHAPTER V
CIRCULAR MEASURE—GRAPHICAL REPRESENTATION

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular Measure</td>
<td>55</td>
</tr>
<tr>
<td>Periodicity of the Trigonometric Functions</td>
<td>57</td>
</tr>
<tr>
<td>Graphical Representation</td>
<td>58</td>
</tr>
</tbody>
</table>

CHAPTER VI
COMPUTATION OF LOGARITHMS AND OF THE TRIGONOMETRIC FUNCTIONS—DE MOIVRE'S THEOREM—HYPERBOLIC FUNCTIONS

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental Series</td>
<td>63</td>
</tr>
<tr>
<td>Computation of Logarithms</td>
<td>64</td>
</tr>
<tr>
<td>Computation of Trigonometric Functions</td>
<td>68</td>
</tr>
<tr>
<td>De Moivre's Theorem</td>
<td>70</td>
</tr>
<tr>
<td>The Roots of Unity</td>
<td>72</td>
</tr>
<tr>
<td>The Hyperbolic Functions</td>
<td>73</td>
</tr>
</tbody>
</table>

CHAPTER VII
MISCELLANEOUS EXERCISES

<table>
<thead>
<tr>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relations of Functions</td>
<td>78</td>
</tr>
<tr>
<td>Right Triangles</td>
<td>80</td>
</tr>
<tr>
<td>Isosceles Triangles and Regular Polygons</td>
<td>83</td>
</tr>
<tr>
<td>Trigonometric Identities and Equations</td>
<td>84</td>
</tr>
<tr>
<td>Oblique Triangles</td>
<td>88</td>
</tr>
</tbody>
</table>
### TABLE OF CONTENTS

**SPHERICAL TRIGONOMETRY**

**CHAPTER VIII**

**RIGHT AND QUADRANTAL TRIANGLES**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivation of Formulas for Right Triangles</td>
<td>93</td>
</tr>
<tr>
<td>Napier's Rules</td>
<td>94</td>
</tr>
<tr>
<td>Ambiguous Case</td>
<td>97</td>
</tr>
<tr>
<td>Quadrantal Triangles</td>
<td>98</td>
</tr>
</tbody>
</table>

**CHAPTER IX**

**OBLIQUE-ANGLED TRIANGLES**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derivation of Formulas</td>
<td>100</td>
</tr>
<tr>
<td>Formulas for Logarithmic Computation</td>
<td>101</td>
</tr>
<tr>
<td>The Six Cases and Examples</td>
<td>104</td>
</tr>
<tr>
<td>Ambiguous Cases</td>
<td>106</td>
</tr>
<tr>
<td>Area of the Spherical Triangle</td>
<td>108</td>
</tr>
</tbody>
</table>

**CHAPTER X**

**APPLICATIONS TO THE CELESTIAL AND TERRESTRIAL SPHERES**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Astronomical Problems</td>
<td>110</td>
</tr>
<tr>
<td>Geographical Problems</td>
<td>113</td>
</tr>
</tbody>
</table>

**CHAPTER XI**

**GRAPHICAL SOLUTION OF A SPHERICAL TRIANGLE**

- 115

**CHAPTER XII**

**RECAPITULATION OF FORMULAS**

- 119

**APPENDIX**

**RELATION OF THE PLANE, SPHERICAL, AND PSEUDO-SPHERICAL TRIGONOMETRIES**

- 125

**ANSWERS TO EXERCISES**

- 129
PLANE TRIGONOMETRY

CHAPTER 1

THE TRIGONOMETRIC FUNCTIONS

ANGLES

1. In Trigonometry the size of an angle is measured by the amount one side of the angle has revolved from the position of the other side to reach its final position.

Thus, if the hand of a clock makes one-fourth of a revolution, the angle through which it turns is one right angle; if it makes one-half a revolution, the angle is two right angles; if one revolution, the angle is four right angles; if one and one-half revolutions, the angle is six right angles, etc.

The amount the side $OB$ has rotated from $OA$ to reach its final position may or may not be equal to the inclination of the lines. In Fig. 1 it is equal to this inclination; in Fig. 4 it is not.

Two angles may have the same sides and yet be different. In Fig. 2
and Fig. 4 the positions of the sides of the angles are the same; yet in
Fig. 2 the angle is two right angles, in Fig. 4 it is six right angles. The
addition of any number of complete revolutions to an angle does not change
the position of its sides.

*Question.*—Through how many right angles does the hour-hand
of a clock revolve in $6\frac{1}{2}$ hours? the minute-hand?

*Question.*—If the fly-wheel of an engine makes 100 revolutions per
minute, through how many right angles does it revolve in 1 second?

---

**Def.**—The first side of the angle—that is, the side from
which the revolution is measured—is the initial line; the
second side is the terminal line.

**Def.**—If the direction of the revolution is opposite to that
of the hands of a clock, the angle is positive; if the same
as that of the hands of a clock, the angle is negative.

The angles we have employed as illustrations—those described
by the hands of a clock—are all negative angles.

2. Angles are usually measured in degrees, minutes, and
seconds. A degree is one-ninetieth of a right angle, a min-
ute is one-sixtieth of a degree, a second is one-sixtieth of a
minute.
The symbols indicating degrees, minutes, and seconds are ° ' ″; thus, twenty-six degrees, forty-three minutes, and ten seconds is written 26° 43′ 10″.

3. The plane about the vertex of an angle is divided into four quadrants, as shown in the figure; the first quadrant begins at the initial line.

![Diagram of the four quadrants]

An angle is said to be in a certain quadrant if its terminal line is in that quadrant.

EXERCISES

4. (1.) Express $2\frac{1}{2}$ right angles in degrees, minutes, and seconds. In what quadrant is the angle?

(2.) What angle less than 360° has the same initial and terminal lines as an angle of 745°?

(3.) What positive angles less than 720° have the same sides as an angle of $-73°$?

(4.) In what quadrant is an angle of $-890°$?
DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

5. The trigonometric functions are numbers, and are defined as the ratios of lines.

Let the angle $AOP$ be so placed that the initial line is horizontal, and from $P$, any point of the terminal line, draw $PS$ perpendicular to the initial line.

Denote the angle $AOP$ by $x$.

$$\frac{SP}{OP} = \text{sine of } x \text{ (written } \sin x\text{)}.$$

$$\frac{OS}{OP} = \text{cosine of } x \text{ (written } \cos x\text{)}.$$
\[ \frac{SP}{OS} = \text{tangent of } x \text{ (written } \tan x) \]

\[ \frac{OS}{SP} = \text{cotangent of } x \text{ (written } \cot x) \]

\[ \frac{OP}{OS} = \text{secant of } x \text{ (written } \sec x) \]

\[ \frac{OP}{SP} = \text{cosecant of } x \text{ (written } \csc x) \]

To the above may be added the versed sine (written versin) and covered sine (written coversin), which are defined as follows:

\[ \text{versin } x = 1 - \cos x; \quad \text{coversin } x = 1 - \sin x. \]

The values of the sine, cosine, etc., do not depend upon what point of the terminal line is taken as \( P \), but upon the angle.

For the triangles \( OSP \) and \( OS'P' \) being similar, the ratio of any two sides of \( OS'P' \) is equal to the ratio of the corresponding sides of \( OSP \).

Def.—The sine, cosine, tangent, cotangent, secant, and cosecant of an angle are the trigonometric functions of the angle, and depend for their value on the angle alone.

6. A line may by its length and direction represent a number; the magnitude of the number is expressed by the length of the line; the number is positive or negative according to the direction of the line.
7. In § 5, if the denominators of the several ratios be taken equal to unity, the trigonometric functions will be represented by lines.

Thus, \( \sin x = \frac{SI}{OP} = \frac{SP}{1} = SP \) = the number represented by the line, that is, the ratio of the line to its unit of length.

Hence \( SI \) may represent the sine of \( x \).

In a similar manner the other trigonometric functions may be represented by lines.

In the following figures a circle of unit radius is described about the vertex \( O \) of the angle \( AOP \), this angle being denoted by \( x \). Then from § 5 it follows that
THE TRIGONOMETRIC FUNCTIONS

SP represents the sine of x.
OS represents the cosine of x.
AT represents the tangent of x.
BC represents the cotangent of x.
OT represents the secant of x.
OC represents the cosecant of x.

For the sake of brevity, the lines SP, OS, etc., of the preceding figures are often spoken of as the sine, cosine, etc.

Hence, we may also define the trigonometric functions in general terms as follows:

If a circle of unit radius is described about the vertex of an angle,

1. The sine of the angle is represented by the perpendicular upon the initial line from the intersection of the terminal line with the circumference.

2. The cosine of the angle is represented by the segment of the initial line extending from the vertex to the sine.

3. The tangent of the angle is represented by a line tangent to the circle at the beginning of the first quadrant, and extending from the point of tangency to the terminal line.

4. The cotangent of the angle is represented by a line tangent to the circle at the beginning of the second quadrant, and extending from the point of tangency to the terminal line.

5. The secant of the angle is represented by the segment of the terminal line extending from the vertex to the tangent.

6. The cosecant of the angle is represented by the segment of the terminal line extending from the vertex to the cotangent.

The definitions in § 5 are called the ratio definitions of the trigonometric functions, and those in § 7 the line definitions. The introduction of two definitions for the same thing should not embarrass the student. We have shown that they are equivalent. In some cases it is convenient to use the first definition, and in other cases the second, as the student will observe in the course of this study. It is therefore important that he should become familiar with the use of both.
SIGNS OF THE TRIGONOMETRIC FUNCTIONS

8. Lines are regarded as positive or negative according to their directions. Thus, in the figures of § 5, OP is positive if it extends to the right of O along the initial line, negative if it extends to the left; SP is positive if it extends upward from OA, negative if it extends downward. OP, the terminal line, is always positive.

The above determines, from § 5, the signs of the trigonometric functions, since it shows the signs of the two terms of each ratio.

By the line definitions the signs may be determined directly. The sine and tangent are positive if measured upward from OA, and negative if measured downward.

The cosine and cotangent are positive if measured to the right from OB, and negative if measured to the left.
The secant and cosecant are positive if measured in the same direction as the terminal line, OP; negative if measured in the opposite direction.

The signs of the functions of angles in the different quadrants are as follows:

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine and cosecant</td>
<td>+</td>
<td>+</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>Cosine and secant</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>Tangent and cotangent</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
</tbody>
</table>

9. It is evident that the values of the functions of an angle depend only upon the position of the sides of the angle. If two angles differ by 360°, or any multiple of 360°, the position of the sides is the same, hence the values of the functions are the same.

Thus in Fig. 1 the angle is 120°, in Fig. 2 the angle is 840°, yet the lines which represent the functions are the same for both angles.

EXERCISE

Determine, by drawing the necessary figures, the sign of tan 1000°; cos 810°; sin 760°; cot −70°; cos −550°; tan −560°; sec 300°; cot 1560°; sin 130°; cos 260°; tan 310°.
RELATIONS OF THE FUNCTIONS

10. By § 5, whatever may be the length of $OP$, we have

$$\frac{SP}{OP} = \sin x; \quad \frac{OS}{OP} = \cos x; \quad \frac{SP}{SP} = \tan x; \quad \frac{OS}{SP} = \cot x; \quad \frac{OP}{OS} = \sec x; \quad \frac{OP}{SP} = \csc x.$$

![Fig. 1](image1.png) ![Fig. 2](image2.png) ![Fig. 3](image3.png)

We have, then, from Figs. 2 and 3,

$$\frac{SP}{OS} = \tan x = \frac{\sin x}{\cos x}; \quad \frac{OS}{SP} = \cot x = \frac{\cos x}{\sin x}.$$  \hspace{1cm} (1)

Multiplying (1) by (2),

$$\tan x \cot x = 1,$$  \hspace{1cm} (3)

or

$$\tan x = \frac{1}{\cot x}; \quad \cot x = \frac{1}{\tan x}.$$  \hspace{1cm} (4)

Again, from Figs. 2 and 3,

$$\frac{OP}{OS} = \sec x = \frac{1}{\cos x}; \quad \frac{OP}{SP} = \csc x = \frac{1}{\sin x}.$$  \hspace{1cm} (5)

From Figs. 2 and 3, $OS^2 + SP^2 = OP^2$,

or

$$\sin^2 x + \cos^2 x = 1,$$  \hspace{1cm} (6)

and

$$\sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x.$$  \hspace{1cm} (7)

Also, $OA^2 + AT^2 = OT^2$, and $OB^2 + BC^2 = OC^2$,

or

$$1 + \tan^2 x = \sec^2 x; \quad 1 + \cot^2 x = \csc^2 x.$$  \hspace{1cm} (8)
The angle $x$ has been taken in the first quadrant; the results are, however, true for any angle. The proof is the same for angles in other quadrants, except that $SP$ becomes negative in the third and fourth quadrants, and $OS$ in the second and third.

**EXERCISES**

11.  
(1.) Prove $\cos x \sec x = 1$.  
(2.) Prove $\sin x \csc x = 1$.  
(3.) Prove $\tan x \cos x = \sin x$.  
(4.) Prove $\sin x \sqrt{1 - \cos^2 x} = 1 - \cos^2 x$.  
(5.) Prove $\tan x + \cot x = \frac{1}{\sin x \cos x}$.  
(6.) Prove $\sin^4 x - \cos^4 x = 1 - 2 \cos^2 x$.  
(7.) Prove $\frac{1}{\cot x \sec x} = \sin x$.  
(8.) Prove $\tan x \sin x + \cos x = \sec x$.

12. The formulas (1)–(8) of § 10 are algebraic equations connecting the different functions of the same angle. If the value of one of the functions of an angle is given, we can substitute this value in one of the equations and solve to find another of the functions. Repeating the process, we find a third function, etc.

In solving equation (6), (7), or (8) a square root is extracted; unless something is given which determines whether to choose the positive or negative square root, we get two values for some of the functions. The reason for this is that there are two angles, less than $360^\circ$ for which a function has a given value.

**EXERCISES**

13. (1.) Given $x$ less than $90^\circ$ and $\sin x = \frac{1}{2}$; find all the other functions of $x$.

*Solution.*

$$\cos x = \pm \sqrt{1 - \frac{1}{4}} = \pm \frac{1}{2}\sqrt{3}.$$  
Since $x$ is less than $90^\circ$, we know that $\cos x$ is positive.
Hence
\[
\begin{align*}
\cos x &= +\frac{1}{2}\sqrt{3}; \\
\tan x &= -\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}; \\
\cot x &= \frac{1}{\frac{1}{2}\sqrt{3}} = \sqrt{3}; \\
\sec x &= \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{3}{2}\sqrt{3}; \\
\csc x &= \frac{1}{\frac{1}{3}} = 2.
\end{align*}
\]

(2.) Given \(\tan x = -\frac{1}{3}\) and \(x\) in quadrant IV; find \(\sin x\) and \(\cos x\).

Solution.—

\[
\frac{\sin x}{\cos x} = -\frac{1}{3};
\]

hence

\[
3 \sin x = -\cos x,
\]

\[
\sin^2 x + \cos^2 x = 1;
\]

hence

\[
10 \sin^2 x = 1;
\]

\[
\sin x = -\sqrt{\frac{1}{10}} = -\frac{\sqrt{10}}{10};
\]

\[
\cos x = \frac{\sqrt{10}}{10}.
\]

(3.) Given \(\sin(-30^\circ) = -\frac{1}{2}\); find the other functions of \(-30^\circ\).

(4.) Given \(x\) in quadrant III and \(\sin x = -\frac{1}{3}\); find all the other functions of \(x\).

(5.) Given \(y\) in quadrant IV and \(\sin y = -\frac{2}{3}\); find all the other functions of \(y\).

(6.) Given \(\cos 60^\circ = \frac{1}{2}\); find all the other functions of \(60^\circ\).

(7.) Given \(\sin 0^\circ = 0\); find \(\cos 0^\circ\) and \(\tan 0^\circ\).

(8.) Given \(\tan z = \frac{1}{3}\) and \(z\) in quadrant I; find the other functions of \(z\).

(9.) Given \(\cot 45^\circ = 1\); find all the other functions of \(45^\circ\).

(10.) Given \(\tan y = \frac{1}{2}\sqrt{5}\) and \(\cos y\) negative; find all the other functions of \(y\).

(11.) Given \(\cot 30^\circ = \sqrt{3}\); find the other functions of \(30^\circ\).

(12.) Given \(2 \sin x = 1 - \cos x\) and \(x\) in quadrant II; find \(\sin x\) and \(\cos x\).

(13.) Given \(\tan x + \cot x = 3\) and \(x\) in quadrant I; find \(\sin x\).
FUNCTIONS OF AN ACUTE ANGLE OF A RIGHT TRIANGLE

14. The functions of an acute angle of a right triangle can be expressed as ratios of the sides of the triangle.

\[ \sin A = \frac{BC}{AB} = \frac{a}{c} = \cos B; \]
\[ \cos A = \frac{AC}{AB} = \frac{b}{c} = \sin B; \]
\[ \tan A = \frac{BC}{AC} = \frac{a}{b} = \cot B; \]
\[ \cot A = \frac{AC}{BC} = \frac{b}{a} = \tan B. \]

15. From § 14, for an acute angle of a right triangle, we have

\[ \text{sine} = \frac{\text{side opposite angle}}{\text{hypotenuse}}; \]
\[ \text{cosine} = \frac{\text{side adjacent to angle}}{\text{hypotenuse}}; \]
\[ \text{tangent} = \frac{\text{side opposite angle}}{\text{side adjacent to angle}}; \]
\[ \text{cotangent} = \frac{\text{side adjacent to angle}}{\text{side opposite angle}}. \]
FUNCTIONS OF COMPLEMENTARY ANGLES

16. From § 14, we have

\[
\begin{align*}
\sin A &= \cos B = \cos(90^\circ - A); \\
\cos A &= \sin B = \sin(90^\circ - A); \\
\tan A &= \cot B = \cot(90^\circ - A); \\
\cot A &= \tan B = \tan(90^\circ - A).
\end{align*}
\]

(9)

Because of this relation the sine and cosine are called co-functions of each other, and the tangent and cotangent are called co-functions of each other.

The results of this article may be stated thus:

A function of an acute angle is equal to the co-function of its complementary angle.

The values of the functions of the different angles are given in "Trigonometric Tables." By the use of the principle just proved, each function of an angle between 45° and 90° can be found as a function of an angle less than 45°. Consequently, the tables need to be constructed for angles up to 45° only. The tables are so arranged that a number in them can be read either as a function of an angle less than 45° or as the co-function of the complement of this angle.

EXERCISES

17. (i.) Express as functions of an angle less than 45°:

\[
\begin{align*}
\sin 70^\circ; & \quad \cos 89^\circ 30'; & \quad \tan 63^\circ; \\
\cos 60^\circ; & \quad \cot 47^\circ; & \quad \sin 72^\circ 39'.
\end{align*}
\]

(2.) \(\cos x = \sin 2x\); find \(x\).

(3.) \(\tan x = \cot 3x\); find \(x\).

(4.) \(\sin 2x = \cos 3x\); find \(x\).

(5.) \(\cot(30^\circ - x) = \tan(30^\circ + 3x)\); find \(x\).

(6.) \(A, B,\) and \(C\) are the angles of a triangle; prove that

\[\cos \frac{1}{2} B = \sin \frac{1}{2}(A + C).\]

Hint.— \(A + B + C = 180^\circ\).
FUNCTIONS OF \(0^\circ, 90^\circ, 180^\circ, 270^\circ, \text{ AND } 360^\circ\)

18. As the angle \(x\) decreases towards \(0^\circ\) (Fig. 1), \(\sin x\) decreases and \(\cos x\) increases. When \(OP\) comes into coincidence with \(OA\), \(SP\) becomes \(0\), and \(OS\) becomes \(OA(=1)\).

Hence \(\sin 0^\circ = 0, \cos 0^\circ = 1\).

As the angle \(x\) increases towards \(90^\circ\) (Fig. 2), \(\sin x\) increases and \(\cos x\) decreases. When \(OP\) comes into coincidence with \(OB\), \(SP\) becomes \(OB(=1)\) and \(OS\) becomes \(0\).

Hence \(\sin 90^\circ = 1, \cos 90^\circ = 0\).

As the angle \(x\) decreases towards \(0^\circ\) (Fig. 3), \(\tan x\) decreases and \(\cot x\) increases. When \(OP\) comes into coincidence with \(OA\), \(AT\) becomes \(0\) and \(BC\) has increased without limit.

Hence \(\tan 0^\circ = 0, \cot 0^\circ = \infty\).

As the angle \(x\) increases towards \(90^\circ\) (Fig. 4), \(\tan x\) increases and \(\cot x\) decreases. When \(OP\) comes into coincidence with \(OB\), \(AT\) has increased without limit, and \(BC = 0\).

Hence \(\tan 90^\circ = \infty, \cot 90^\circ = 0\).

Remark.—By \(\cot 0^\circ = \infty\) we mean that as the angle approaches indefinitely near to \(0^\circ\) its cotangent increases so as to become greater than any finite quantity we may choose. The symbol \(\infty\) does not denote a definite number, but simply that the number is indefinitely great.
In every case where a trigonometric function becomes indefinitely great it is in a positive sense if the angle approaches the limiting value from one side, in a negative sense if the angle approaches the limiting value from the other side. Thus \( \cot 0^\circ = +\infty \) if the angle decreases to \( 0^\circ \), but \( \cot 0^\circ = -\infty \) if the angle increases from a negative angle to \( 0^\circ \). We shall not often need to distinguish between \( +\infty \) and \( -\infty \), and shall in general denote either by the symbol \( \infty \).

By a similar method the functions of \( 180^\circ \), \( 270^\circ \), and \( 360^\circ \) may be deduced. The results of this article are shown in the following table:

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \cos )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \tan )</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
</tr>
<tr>
<td>( \cot )</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

19. It may now be stated that, as an angle varies, its sine and cosine can take on values from \(-1\) to \(+1\) only, its tangent and cotangent all values from \(-\infty\) to \(+\infty\), its secant and cosecant all values from \(-\infty\) to \(+\infty\), except those between \(-1\) and \(+1\).

FUNCTIONS OF THE SUPPLEMENT OF AN ANGLE

20. Suppose the triangle \( OPS \) (Fig. 1) equal to the triangle \( OP'S' \) (Fig. 2), then \( SP=S'P' \) and \( OS=OS' \), and the angle \( AOP' \) (Fig. 2) is equal to the supplement of \( AOP \) (Fig. 1). Also, in the triangle \( AOP' \) (Fig. 3), angle \( AOP' = \text{angle } AOP' \) (Fig. 2).
THE TRIGONOMETRIC FUNCTIONS

It follows from §§ 5 and 8 that

\[
\begin{align*}
\sin (180^\circ - x) &= \sin x; \\
\cos (180^\circ - x) &= -\cos x; \\
\tan (180^\circ - x) &= -\tan x; \\
\cot (180^\circ - x) &= -\cot x.
\end{align*}
\] (10)

The results of this article may be stated thus:

The sine of an angle is equal to the sine of its supplement, and the cosine, tangent, and cotangent are each equal to minus the same functions of its supplement.

The principle just proved is of great importance in the solution of triangles which contain an obtuse angle.

FUNCTIONS OF 45°, 30°, AND 60°

21. In the right triangle OSP (Fig. 1) angle O = angle P = 45°, and OP = 1.

Hence

\[OS = SP = \frac{1}{\sqrt{2}}.\]

Therefore

\[
\begin{align*}
\sin 45^\circ &= \cos 45^\circ = \frac{1}{\sqrt{2}}; \\
\tan 45^\circ &= \cot 45^\circ = 1.
\end{align*}
\] §14, 16

![Fig. 1](image)

In equilateral triangle OPA (Fig. 2) the sides are of unit length, PS bisects angle OPA, is perpendicular to OA, and bisects OA.

Hence, in the right triangle OPS, \(OS = \frac{1}{2}, SP = \frac{1}{2} \sqrt{3}\).

Therefore

\[
\begin{align*}
\sin 30^\circ &= \cos 60^\circ = \frac{1}{2}; \\
\cos 30^\circ &= \sin 60^\circ = \frac{\sqrt{3}}{2}; \\
\tan 30^\circ &= \cot 60^\circ = \frac{1}{\sqrt{3}}; \\
\cot 30^\circ &= \tan 60^\circ = \sqrt{3}.
\end{align*}
\] §14
22. The following values should be remembered:

<table>
<thead>
<tr>
<th>Angle</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>1</td>
</tr>
<tr>
<td>cos</td>
<td>1</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
</tr>
</tbody>
</table>

EXERCISES

Prove that if \(x = 30°\),

1. \(\sin 2x = 2 \sin x \cos x\);
2. \(\cos 3x = 4 \cos^3 x - 3 \cos x\);
3. \(\cos 2x = \cos^2 x - \sin^2 x\);
4. \(\sin 3x = 3 \sin x \cos^2 x - \sin^3 x\);
5. \(\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}\).

(6.) Prove that the equations of exercises 1 and 3 are correct if \(x = 45°\).

(7) Prove that the equations of exercises (2) and (4) are correct if \(x = 120°\).

The following three articles, §§ 23–25, are inserted for completeness. They include the functions of \((90° - x)\) and \((180° - x)\), which, on account of their great importance, were treated separately in §§ 16 and 20.

FUNCTIONS OF \((-x)\), \((180° - x)\), \((180° + x)\), \((360° - x)\)

23. The line representing any function—as sine, cosine, etc.—of each of these angles has the same length as the line representing the same function of \(x\).

Thus in Figs. 2 and 3, triangle \(OS'P' = \triangle OSP\), hence \(SP = S'P'\), and \(OS = OS'\).
In Figs. 1 and 4, triangle $OSP'$ = triangle $OSP$, hence $SP' = SP$.
In Figs. 1, 2, and 4, triangle $OAT'' = triangle OAT$, hence $AT'' = AT$.
In Figs. 1, 2, and 4, triangle $OBC' = triangle OBC$, hence $BC' = BC$.

Therefore any function of each of the angles $(−x)$, $(180° − x)$, $(180° + x)$, $(360° − x)$, is equal in numerical value to the same function of $x$. Its sign, however, depends on the direction of the line representing it.

Putting in the correct sign, we obtain the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>$−x$</th>
<th>$(180° − x)$</th>
<th>$(180° + x)$</th>
<th>$(360° − x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin (−x)$</td>
<td>$−\sin x$</td>
<td>$\sin (180° − x)$</td>
<td>$−\sin x$</td>
<td>$\sin (360° − x)$</td>
</tr>
<tr>
<td>$\cos (−x)$</td>
<td>$\cos x$</td>
<td>$−\cos x$</td>
<td>$\cos (180° − x)$</td>
<td>$\cos (360° − x)$</td>
</tr>
<tr>
<td>$\tan (−x)$</td>
<td>$−\tan x$</td>
<td>$−\tan x$</td>
<td>$\tan (180° − x)$</td>
<td>$−\tan x$</td>
</tr>
<tr>
<td>$\cot (−x)$</td>
<td>$−\cot x$</td>
<td>$−\cot x$</td>
<td>$\cot (180° − x)$</td>
<td>$−\cot x$</td>
</tr>
<tr>
<td>$\sin (180° + x)$</td>
<td>$−\sin x$</td>
<td></td>
<td>$\sin (360° − x)$</td>
<td></td>
</tr>
<tr>
<td>$\cos (180° + x)$</td>
<td>$−\cos x$</td>
<td></td>
<td>$\cos (360° − x)$</td>
<td></td>
</tr>
<tr>
<td>$\tan (180° + x)$</td>
<td>$\tan x$</td>
<td></td>
<td>$\tan (360° − x)$</td>
<td></td>
</tr>
<tr>
<td>$\cot (180° + x)$</td>
<td>$\cot x$</td>
<td></td>
<td>$\cot (360° − x)$</td>
<td></td>
</tr>
</tbody>
</table>
FUNCTIONS OF \((90° - y), (90° + y), (270° - y), (270° + y)\)

24. The line representing the sine of each of these angles is of the same length as the line representing the cosine of \(y\); the cosine, tangent, or cotangent, respectively, are of the same length as the sine, cotangent, and tangent of \(y\).

For

Triangle \(OS'P' = \triangle OSP\), hence \(S'P' = OS\), and \(OS' = SP\).

Triangle \(OAT' = \triangle OBC\), hence \(AT' = BC\).

Triangle \(OBC' = \triangle OAT\), hence \(BC = AT\).

Therefore any function of each of the angles \((90° - y), (90° + y), (270° - y), (270° + y)\), is equal in numerical value to the co-function
of y. Its sign, however, depends on the direction of the line representing it.

Putting in the correct sign, we obtain the following table:

<table>
<thead>
<tr>
<th>Function</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin (90^\circ - y) )</td>
<td>( \cos y )</td>
</tr>
<tr>
<td>( \cos (90^\circ - y) )</td>
<td>( \sin y )</td>
</tr>
<tr>
<td>( \tan (90^\circ - y) )</td>
<td>( \cot y )</td>
</tr>
<tr>
<td>( \cot (90^\circ - y) )</td>
<td>( \tan y )</td>
</tr>
<tr>
<td>( \sin (270^\circ - y) )</td>
<td>( -\cos y )</td>
</tr>
<tr>
<td>( \cos (270^\circ - y) )</td>
<td>( -\sin y )</td>
</tr>
<tr>
<td>( \tan (270^\circ - y) )</td>
<td>( \cot y )</td>
</tr>
<tr>
<td>( \cot (270^\circ - y) )</td>
<td>( \tan y )</td>
</tr>
</tbody>
</table>

25. Either of the two preceding articles enables us directly to express the functions of any angle, positive or negative, in terms of the functions of a positive angle less than \( 90^\circ \).

Thus,

\[
\begin{align*}
\sin 212^\circ &= \sin (180^\circ + 32^\circ) = -\sin 32^\circ; \\
\cos 260^\circ &= \cos (270^\circ - 10^\circ) = -\sin 10^\circ.
\end{align*}
\]

**EXERCISES**

(I.) What angles less than \( 360^\circ \) have the sine equal to \( -\frac{1}{\sqrt{2}} \)? the tangent equal to \( \sqrt{3} \)?

(2.) For what values of \( x \) less than \( 720^\circ \) is \( \sin x = \frac{1}{\sqrt{3}} \)?

(3.) Find the sine and cosine of \( -30^\circ; 765^\circ; 120^\circ; 210^\circ \).

(4.) Find the functions of \( 405^\circ; 600^\circ; 1125^\circ; -45^\circ; 225^\circ \).

(5.) Find the functions of \( -120^\circ; -225^\circ; -420^\circ; 3270^\circ \).

(6.) Express as functions of an angle less than \( 45^\circ \) the functions of \( 233^\circ; -197^\circ; 894^\circ \).

(7.) Express as functions of an angle between \( 45^\circ \) and \( 90^\circ \), \( \sin 267^\circ; \tan (-254^\circ); \cos 950^\circ \).

(8.) Given \( \cos 164^\circ = -0.96 \), find \( \sin 196^\circ \).

(9.) Simplify \( \cos (90^\circ + x) \cos (270^\circ - x) - \sin (180^\circ - x) \sin (360^\circ - x) \).

(10.) Simplify \( \frac{\sin (180^\circ - x)}{\sin (270^\circ - x)} \tan (90^\circ - x) + \frac{1}{\sin^2 (270^\circ - x)} \).

(11.) Express the functions of \( (x - 90^\circ) \) in terms of functions of \( x \).
CHAPTER II

THE RIGHT TRIANGLE

27. To solve a triangle is to find the parts not given.

A triangle can be solved if three parts, at least one of which is a side, are given. A right triangle has one angle, the right angle, always given; hence a right triangle can be solved if two sides, or one side and an acute angle, are also given.

The parts of the right triangle not given are found by the use of the following formulas:

\[(1) \ \text{sine} = \frac{\text{opposite side}}{\text{hypotenuse}}; \quad (2) \ \text{cosine} = \frac{\text{adjacent side}}{\text{hypotenuse}}; \quad \text{§ 14}\]

\[(3) \ \text{tangent} = \frac{\text{opposite side}}{\text{adjacent side}}; \quad (4) \ \text{cotangent} = \frac{\text{adjacent side}}{\text{opposite side}};\]

\[(5) c^2 = a^2 + b^2; \quad (6) B = (90^\circ - A). \quad \text{§ 16}\]

To solve, select a formula in which two given parts enter; substituting in this the given values, a third part is found. Continue this method till all the parts are found.

In a given problem there are several ways of solving the triangle; choose the shortest.

EXAMPLE

The hypotenuse of a right triangle is 47.653, a side is 21.34; find the remaining parts and the area.

![Diagram of a right triangle with sides labeled a, b, and c, and angle B indicated.]
THE RIGHT TRIANGLE

SOLUTION WITHOUT LOGARITHMS

The functions of angles are given in the table of "Natural Functions."

\[
\sin A = \frac{a}{c} = \frac{21.34}{47.653}
\]

\[
47.653 \times 21.34 = 1000.000 (4.4788) \quad \frac{190612}{227880}
\]

\[
\frac{190612}{312360} \quad \frac{372680}{333571}
\]

\[
\frac{372680}{333571} \quad \frac{391090}{381224} \quad 9866
\]

\[
\sin A = .4478 \quad A = 26^\circ 36'
\]

\[
b = c \cos A
\]

\[
= 47.653 \times .8942
\]

\[
\frac{47.653}{95306} \quad \frac{428877}{381224} \quad 42.6113126
\]

\[
b = 42.61
\]

\[
B = (90^\circ - 26^\circ 36') = 63^\circ 24'
\]

area = \(\frac{1}{2} ab\)

\[
= \frac{1}{2} \times 21.34 \times 42.61
\]

\[
21.34 \quad 42.61
\]

\[
2134 \quad 12804 \quad 4268 \quad 8536
\]

\[
21092.2974 \quad 454.6487
\]

area = 454.6

SOLUTION EMPLOYING LOGARITHMS

It is usually better to solve triangles by the use of logarithms.

The logarithms of the functions are given in the tables of "Logarithms of Functions."

\[
\sin A = \frac{a}{c}
\]

\[
\log \sin A = \log a - \log c
\]

\[
\log 21.34 = 1.32919
\]

\[
\log 47.653 = 1.67809 \text{ sub.}
\]

\[
\log \sin A = 9.65110 - 10
\]

\[
A = 26^\circ 36' 14''
\]

\[
\cos A = \frac{b}{c}
\]

\[
\log \cos A = \log \frac{b}{c} = \log \cos A
\]

\[
\log \cos 26^\circ 36' 14'' = 9.95140 - 10
\]

\[
\log b = 1.62949
\]

\[
b = 42.608
\]

\[
B = (90^\circ - 26^\circ 36' 14'') = 63^\circ 23' 46''
\]

area = \(\frac{1}{2} ab\)

\[
\log \text{area} = \log \frac{1}{2} + \log a + \log b
\]

\[
\log \frac{1}{2} = 9.69897 - 10
\]

\[
\log 21.34 = 1.32919
\]

\[
\log 42.608 = 1.62949
\]

\[
\log \text{area} = 2.65765
\]

area = 454.62

* In this solution the five-place table of the "Logarithms of Functions" is used.

† No more decimal places are retained, because the figures in them are not accurate; this is due to the fact that the table of "Natural Functions" is only four-place.
CHECK ON THE CORRECTNESS OF THE WORK

\[ a^2 = c^2 - b^2 = (c + b)(c - b) \]
\[ = 90.263 \times 5.043 \]
\[ 90.263 \]
\[ 5.043 \]
\[ 270789 \]
\[ 361052 \]
\[ 4513150 \]
\[ a^2 = 455.196309 \]

Extracting the square root, \( a = 21.34 \), which proves the solution correct.

Remark.—The results obtained in the solution of the preceding exercise without logarithms are less accurate than those obtained in the solution by the use of logarithms; the cause of this is that four-place tables have been used in the former method, five place in the latter.

EXERCISES

28. (1.) In a right triangle \( b = 96.42, c = 114.81 \); find \( a \) and \( A \).

(2.) The hypotenuse of a right triangle is 28.453, a side is 18.197; find the remaining parts.

(3.) Given the hypotenuse of a right triangle = 747.24, an acute angle = 23° 45'; find the remaining parts.

(4.) Given a side of a right triangle = 37.234, the angle opposite = 54° 27'; find the remaining parts and the area.

(5.) Given a side of a right triangle = 1.1293, the angle adjacent = 74° 13' 27''; find the remaining parts and the area.

(6.) In a right triangle \( A = 15° 22' 11'' \), \( c = .01793 \); find \( b \).

(7.) In a right triangle \( B = 71° 34' 53'' \), \( b = 896.33 \); find \( a \).

(8.) In a right triangle \( c = 3729.4, b = 2869.1 \); find \( A \).

(9.) In a right triangle \( a = 1247, b = 1988 \); find \( c \).

(10.) In a right triangle \( a = 8.6432, b = 4.7815 \); find \( B \).

The angle of elevation or depression of an object is the angle a line from the point of observation to the object makes with the horizontal.
Thus angle \( x \) (Fig. 1) is the angle of elevation of \( P \) if \( O \) is the point of observation; angle \( y \) (Fig. 2) is the angle of depression of \( P \) if \( O \) is the point of observation.

(11.) At a horizontal distance of 253 ft. from the base of a tower the angle of elevation of the top is \( 60^\circ \ 20' \); find the height of the tower.

(12.) From the top of a vertical cliff 85 ft. high the angle of depression of a buoy is \( 24^\circ \ 31' \ 22'' \); find the distance of the buoy from the foot of the cliff.

(13.) A vertical pole 31 ft. high casts a horizontal shadow 45 ft. long; find the angle of elevation of the sun above the horizon.

(14.) From the top of a tower 115 ft. high the angle of depression of an object on a level road leading away from the tower is \( 22^\circ \ 13' \ 44'' \); find the distance of the object from the top of the tower.

(15.) A rope 324 ft. long is attached to the top of a building, and the inclination of the rope to the horizontal, when taut, is observed to be \( 47^\circ \ 21' \ 17'' \); find the height of the building.

(16.) A light-house is 150 ft. high. How far is an object on the surface of the water visible from the top?

[Take the radius of the earth as 3960 miles.]

(17.) Three buoys are at the vertices of a right triangle; one side of the triangle is 17,894 ft., the angle adjacent to it is \( 57^\circ \ 23' \ 46'' \). Find the length of a course around the three buoys.

(18.) The angle of elevation of the top of a tower observed from a point at a horizontal distance of 897.3 ft. from the base is \( 10^\circ \ 27' \ 42'' \); find the height of the tower.

(19.) A ladder 42 \( \frac{1}{2} \) ft. long leans against the side of a building; its foot is 25 \( \frac{1}{4} \) ft. from the building. What angle does it make with the ground?

(20.) Two buildings are on opposite sides of a street 120 ft. broad.
The height of the first is 55 ft.; the angle of elevation of the top of the second, observed from the edge of the roof of the first, is $26^\circ \ 37'$. Find the height of the second building.

(21.) A mark on a flag-pole is known to be 53 ft. 7 in. above the ground. This mark is observed from a certain point, and its angle of elevation is found to be $25^\circ \ 34'$. The angle of elevation of the top of the pole is then measured, and found to be $34^\circ \ 17'$. Find the height of the pole.

(22.) The equal sides of an isosceles triangle are each 7 in. long; the base is 9 in. long. Find the angles of the triangle.

\[ \text{Hint.} - \text{Draw the perpendicular } BD. \ \text{BD bisects the base, and also the angle } ABC. \]

\text{In the right triangle } ABD, AB=7 \text{ in.}, AD=4\frac{1}{2} \text{ in.}, \text{hence } ABD \text{ can be solved.}

\text{Angle } C=\text{angle } A, \text{angle } ABC=2 \text{ angle } ABD.

(23.) Given the equal sides of an isosceles triangle each 13.44 in. and the equal angles are each $63^\circ \ 21' \ 42''$; find the remaining parts and the area.

(24.) The equal sides of an isosceles triangle are each 377.22 in., the angle between them is $19^\circ \ 55' \ 32''$. Find the base and the area of the triangle.

(25.) If a chord of a circle is 18 ft. long, and it subtends at the centre an angle of $45^\circ \ 31' \ 10''$, find the radius of the circle.

(26.) The base of a wedge is 3.92 in., and its sides are each 13.25 in. long; find the angle at its vertex.
(27.) The angle between the legs of a pair of dividers is $64^\circ 45'$, the legs are 5 in. long; find the distance between the points.

(28.) A field is in the form of an isosceles triangle, the base of the triangle is 1793.2 ft.; the angles adjacent to the base are each $53^\circ 27' 49''$. Find the area of the field.

(29.) A house has a gable roof. The width of the house is 30 ft., the height to the eaves $25\frac{1}{2}$ ft., the height to the ridge-pole $33\frac{1}{2}$ ft. Find the length of the rafters and the area of an end of the house.

(30.) The length of one side of a regular pentagon is 29.25 in.; find the radius, the apothem, and the area of the pentagon.

**Hint.**—The pentagon is divided into 5 equal isosceles triangles by its radii. Let $AOB$ be one of these triangles. $AB=29.25$ in.; angle $AOB=\frac{1}{5}$ of $360^\circ=72^\circ$. Find, by the methods previously given, $OA$, $OD$, and the area of the triangle $AOB$.

These are the radius of the pentagon, the apothem of the pentagon, and \frac{1}{5} the area of the pentagon respectively.

(31.) The apothem of a regular dodecagon is 2; find the perimeter.

(32.) A tower is octagonal; the perimeter of the octagon is 153.7 ft. Find the area of the base of the tower.

(33.) A fence extends about a field which is in the form of a regular polygon of 7 sides; the radius of the polygon is 6283.4 ft. Find the length of the fence.

(34.) The length of a side of a regular hexagon inscribed in a circle is 3.27 ft.; find the perimeter of a regular decagon inscribed in the same circle.

(35.) The area of a field in the form of a regular polygon of 9 sides is 483930 sq. ft.; find the length of the fence about it.
SOLUTION OF OBLIQUE TRIANGLES BY THE AID OF RIGHT TRIANGLES

29. Oblique triangles can always be solved by the aid of right triangles without the use of special formulas; the method is frequently, however, quite awkward; hence, in a later chapter, formulas are deduced which render the solution more simple.

The following exercises illustrate the solution by means of right triangles:

(1.) In an oblique triangle \( a = 3.72 \), \( B = 47^\circ 52' \), \( C = 109^\circ 10' \); find the remaining parts.

*The given parts are a side and two angles.*

\[ \text{Hint.} - A = [180^\circ - (B + C)]. \]

Draw the perpendicular \( CD \).
Solve the right triangle \( BCD \).
Having thus found \( CD \), solve the right triangle \( ACD \).

(2.) In an oblique triangle \( a = 89.7 \), \( c = 125.3 \), \( B = 39^\circ 8' \); find the remaining parts.

*The given parts are two sides and the included angle.*
**THE RIGHT TRIANGLE**

*Hint.*—Draw the perpendicular $CD$.
Solve the right triangle $CBD$.
Having thus found $CD$ and $AD (= c - DB)$, solve the right triangle $ACD$.

(3.) In an oblique triangle $a = 3.67, b = 5.81, A = 27° 23'$; find the remaining parts.

The given parts are two sides and an angle opposite one of them.

Either of the triangles $ACB, ACB'$ contains the given parts, and is a solution.

There are two solutions when the side opposite the given angle is less than the other given side and greater than the perpendicular, $CD$, from the extremity of that side to the base.*

*Hint.*—Solve the right triangle $ACD$.
Having thus found $CD$, solve the right triangle $CDB$ (or $CDB'$).

(4.) The sides of an oblique triangle are $a = 34.2, b = 38.6, c = 55.12$; find the angles.

The given parts are the three sides.

* A discussion of this case is contained in a later chapter on the solution of oblique triangles.
Hint.— Let $DB = x$,

$$
\phi - x^2 = CD^2 = b^2 - (c - x)^2.
$$

Hence

$$
a^2 = b^2 - c^2 + 2cx,
$$

$$
x = \frac{a^2 + c^2 - b^2}{2c}
$$

In each of the right triangles $ACD$ and $BCD$ the hypotenuse and a side are now known; hence these triangles can be solved.

(5.) Two trees, $A$ and $B$, are on opposite sides of a pond. The distance of $A$ from a point $C$ is 297.6 ft., the distance of $B$ from $C$ is 864.4 ft., the angle $ACB$ is $87^\circ\ 43'\ 12''$. Find the distance $AB$.

(6.) To determine the distance of a ship $A$ from a point $B$ on shore, a line, $BC$, 800 ft. long, is measured on shore; the angles, $ABC$ and $ACB$, are found to be $67^\circ\ 43'$ and $74^\circ\ 21'\ 16''$ respectively. What is the distance of the ship from the point $B$?

(7.) A light-house 92 ft. high stands on top of a hill; the distance from its base to a point at the water's edge is 297.25 ft.; observed from this point the angle of elevation of the top is $46^\circ\ 33'\ 15''$. Find the length of a line from the top of the light-house to the point.

(8.) The sides of a triangular field are 534 ft., 679.47 ft., 474.5 ft. What are the angles and the area of the field?

(9.) A certain point is at a horizontal distance of 117\(\frac{1}{2}\) ft. from a river, and is 11 ft. above the river; observed from this point the angle of depression of the farther bank is $1^\circ\ 12'$. What is the width of the river?

(10.) In a quadrilateral $ABCD$, $AB = 1.41$, $BC = 1.05$, $CD = 1.76$, $DA = 1.93$, angle $A = 75^\circ\ 21'$; find the other angles of the quadrilateral.
Hint.—Draw the diagonal $DB$.
In the triangle $ABD$ two sides and an included angle are given, hence the triangle can be solved.
The solution of triangle $ABD$ gives $DB$.
Having found $DB$, there are three sides of the triangle $DBC$ known, hence the triangle can be solved.

(II.) In a quadrilateral $ABCD$, $AB = 12.1$, $AD = 9.7$, angle $A = 47^\circ 18'$, angle $B = 64^\circ 49'$, angle $D = 100^\circ$; find the remaining sides.

Hint.—Solve triangle $ABD$ to find $BD$. 
CHAPTER III
TRIGONOMETRIC ANALYSIS

30. In this chapter we shall prove the following fundamental formulas, and shall derive other important formulas from them:

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y, \]  \hspace{1cm} (11)  
\[ \sin(x - y) = \sin x \cos y - \cos x \sin y, \]  \hspace{1cm} (12)  
\[ \cos(x + y) = \cos x \cos y - \sin x \sin y, \]  \hspace{1cm} (13)  
\[ \cos(x - y) = \cos x \cos y + \sin x \sin y; \]  \hspace{1cm} (14)

PROOF OF FORMULAS (11)–(14)

31. Let angle \( \angle AOQ = x \), angle \( \angle QOP = y \); then angle \( \angle AOP = (x + y) \).

The angles \( x \) and \( y \) are each acute and positive, and in Fig. 1 \( (x + y) \) is less than \( 90^\circ \), in Fig. 2 \( (x + y) \) is greater than \( 90^\circ \).

In both figures the circle is a unit circle, and \( SP \) is perpendicular to \( OA \); hence \( SP = \sin(x + y) \), \( OS = \cos(x + y) \).
TRIGONOMETRIC ANALYSIS

Draw $DP$ perpendicular to $OQ$;
then $DP = \sin y$, $OD = \cos y$,
angle $SPD = \text{angle } AOQ = x$.
(Their sides being perpendicular.)

Draw $DE$ perpendicular to $OA$, $DH$ perpendicular to $SP$.

$\sin(x + y) = SP = ED + HP$.
$ED = (\sin x) \times OD = \sin x \cos y$.
(For $OED$ being a right triangle, $\frac{ED}{OD} = \sin x$.)

$HP = (\cos x) \times DP = \cos x \sin y$.
(For $HPD$ being a right triangle, $\frac{HP}{DP} = \cos x$.)

Therefore, $\sin(x + y) = \sin x \cos y + \cos x \sin y$.  \hspace{1cm} (11)

$\cos(x + y) = OS = OE - HD$.*
$OE = (\cos x) \times OD = \cos x \cos y$.
(For $OED$ being a right triangle, $\frac{OE}{OD} = \cos x$.)

$HD = (\sin x) \times DP = \sin x \sin y$.
(For $PHD$ being a right triangle, $\frac{HD}{DP} = \sin x$.)

Therefore, $\cos(x + y) = \cos x \cos y - \sin x \sin y$.  \hspace{1cm} (13)

32. The preceding formulas have been proved only for the case when $x$ and $y$ are each acute and positive. The proof can, however, readily be extended to include all values of $x$ and $y$.

Let $y$ be acute, and let $x$ be an angle in the second quadrant, then $x = (90^\circ + x')$ where $x'$ is acute.

$\sin(x + y) = \sin(90^\circ + x' + y) = \cos(x' + y)$  \hspace{1cm} \$24$
$= \cos x' \cos y - \sin x' \sin y$
$= \sin(90^\circ + x') \cos y + \cos(90^\circ + x') \sin y$  \hspace{1cm} \$24$
$= \sin x \cos y + \cos x \sin y$.

* If $(x + y)$ is greater than $90^\circ$, $OS$ is negative.
Thus the formula has been extended to the case where one of the angles is obtuse and less than 180°. In a similar way the formula for \( \cos(x+y) \) is extended to this case.

By continuing this method both formulas are proved to be true for all positive values of \( x \) and \( y \).

Any negative angle \( y \) is equal to a positive angle \( y' \), minus some multiple of 360°. The functions of \( y \) are equal to those of \( y' \), and the functions of \( (x+y) \) are equal to those of \( (x+y') \).

Therefore, the formulas being true for \( (x+y') \), are true for \( (x+y) \).

A repetition of this reasoning shows that the formulas are true when both angles, \( x \) and \( y \), are negative.

33. Substituting the angle \( -y \) for \( y \) in formula (11), it becomes

\[
\sin(x - y) = \sin x \cos(-y) + \cos x \sin(-y).
\]

But \( \cos(-y) = \cos y \), and \( \sin(-y) = -\sin y \). § 23

Therefore, \( \sin(x - y) = \sin x \cos y - \cos x \sin y \). \hspace{1cm} (12)

Substituting \( -y \) for \( y \) in formula (13), it becomes

\[
\cos(x - y) = \cos x \cos(-y) - \sin x \sin(-y),
\]

\[= \cos x \cos y + \sin x \sin y.\]

Therefore, \( \cos(x - y) = \cos x \cos y + \sin x \sin y \).* \hspace{1cm} (14)

EXERCISES

34. (i.) Prove geometrically where \( x \) and \( y \) are acute and positive:

\[
\sin(x - y) = \sin x \cos y - \cos x \sin y,
\]

\[
\cos(x - y) = \cos x \cos y + \sin x \sin y.
\]

* Formulas (12) and (14) are proved geometrically in § 34. The geometric proof is complicated by the fact that \( OD \) and \( DP \) are functions of \( -y \), while the functions of \( y \) are what we use.
Hint.—Angle $A O Q = x$, angle $P O Q = y$, and angle $A O P = (x - y)$.

Draw $P D$ perpendicular to $O Q$.

Then $D P = \sin (-y) = -\sin y$; but $D P$ is negative, therefore $P D$ taken as positive is equal to $\sin y$:

$$OD=\cos(-y)=\cos y.$$  

Angle $H P D = \angle A O Q = x$, their sides being perpendicular.

Draw $D H$ perpendicular to $A P$, $D E$ perpendicular to $O A$.

$$\sin (x - y) = SP = ED - PH.$$  

From right triangle $O E D$, $E D = (\sin x) \times O D = \sin x \cos y$.

From right triangle $D H P$, $P H = (\cos x) \times P D = \cos x \sin y$.

Therefore, $\sin (x - y) = \sin x \cos y - \cos x \sin y$.

$$\cos (x - y) = O S = O F + D H.$$  

From right triangle $O E D$, $O E = (\cos x) \times O D = \cos x \cos y$.

From right triangle $D H P$, $D H = (\sin x) \times P D = \sin x \sin y$.

Therefore, $\cos (x - y) = \cos x \cos y + \sin x \sin y$.

(2.) Find the sine and cosine of $(45^\circ + x)$, $(30^\circ - x)$, $(60^\circ + x)$, in terms of $\sin x$ and $\cos x$.

(3.) Given $\sin x = \frac{3}{5}$, $\sin y = \frac{1}{3}$, $x$ and $y$ acute; find $\sin (x + y)$ and $\sin (x - y)$.

(4.) Find the sine and cosine of $75^\circ$ from the functions of $30^\circ$ and $45^\circ$.

Hint. — $75^\circ = (45^\circ + 30^\circ)$.

(5.) Find the sine and cosine of $15^\circ$ from the functions of $30^\circ$ and $45^\circ$.

(6.) Given $x$ and $y$, each in the second quadrant, $\sin x = \frac{1}{2}$, $\sin y = \frac{1}{4}$; find $\sin (x + y)$ and $\cos (x - y)$.

(7.) By means of the above formulas express the sine and cosine of $(180^\circ - x)$, $(180^\circ + x)$, $(270^\circ - x)$, $(270^\circ + x)$, in terms of $\sin x$ and $\cos x$.

(8.) Prove $\sin (60^\circ + 45^\circ) + \cos (60^\circ + 45^\circ) = \cos 45^\circ$.

(9.) Given $\sin 45^\circ = \frac{1}{2} \sqrt{2}$, $\cos 45^\circ = \frac{1}{2} \sqrt{2}$; find $\sin 90^\circ$ and $\cos 90^\circ$.

(10.) Prove that $\sin (60^\circ + x) - \sin (60^\circ - x) = \sin x$. 
TAN
GENT OF THE SUM AND DIFFERENCE OF TWO ANGLES

35. \( \tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} \).

Dividing each term of both numerator and denominator of the right-hand side of this equation by \( \cos x \cos y \), and remembering that \( \frac{\sin}{\cos} = \tan \), we have

\[
\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.
\]

(15)

In a similar way, dividing formula (12) by formula (14), we obtain

\[
\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.
\]

(16)

FUNCTIONS OF TWICE AN ANGLE

36. An important special case of formulas (11), (13), and (15) is when \( y = x \); we then obtain the functions of \( 2x \) in terms of the functions of \( x \).

From (11), \( \sin(x + x) = \sin x \cos x + \cos x \sin x \).

Hence \( \sin 2x = 2 \sin x \cos x \).

(17)

From (13), \( \cos 2x = \cos^2 x - \sin^2 x \).

(18)

Since \( \cos^2 x = 1 - \sin^2 x \), and \( \sin^2 x = 1 - \cos^2 x \), we derive from equation (18),

\( \cos 2x = 1 - 2 \sin^2 x \),

(19)

and \( \cos 2x = 2 \cos^2 x - 1 \).

(20)

From (15), \( \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \).

(21)

FUNCTIONS OF HALF AN ANGLE

37. Equations (19) and (20) are true for any angle; therefore for the angle \( \frac{1}{2} x \).

From (19), \( \cos x = 1 - 2 \sin^2 \frac{1}{2} x \).
TRIGONOMETRIC ANALYSIS

or

\[ \sin^2 \frac{1}{2} x = \frac{1 - \cos x}{2} ; \]

therefore,

\[ \sin \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{2}} . \]  \hspace{1cm} (22)

From (20),
\[ \cos x = 2 \cos^2 \frac{1}{2} x - 1 ; \]

or

\[ \cos^2 \frac{1}{2} x = \frac{1 + \cos x}{2} ; \]

therefore,

\[ \cos \frac{1}{2} x = \pm \sqrt{\frac{1 + \cos x}{2}} . \]  \hspace{1cm} (23)

Dividing (22) by (23), we obtain

\[ \tan \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} . \]  \hspace{1cm} (24)

FORMULAS FOR SUMS AND DIFFERENCES OF FUNCTIONS

38. From formulas (11)–(14), we obtain

\[ \sin (x + y) + \sin (x - y) = 2 \sin x \cos y ; \]
\[ \sin (x + y) - \sin (x - y) = 2 \cos x \sin y ; \]
\[ \cos (x + y) + \cos (x - y) = 2 \cos x \cos y ; \]
\[ \cos (x + y) - \cos (x - y) = -2 \sin x \sin y . \]

Let
\[ u = (x + y) \text{ and } v = (x - y) ; \]
then
\[ x = \frac{1}{2} (u + v), \ y = \frac{1}{2} (u - v) . \]

Substituting in the above equations, we obtain

\[ \sin u + \sin v = 2 \sin \frac{1}{2} (u + v) \cos \frac{1}{2} (u - v) ; \]  \hspace{1cm} (25)
\[ \sin u - \sin v = 2 \cos \frac{1}{2} (u + v) \sin \frac{1}{2} (u - v) ; \]  \hspace{1cm} (26)
\[ \cos u + \cos v = 2 \cos \frac{1}{2} (u + v) \cos \frac{1}{2} (u - v) ; \]  \hspace{1cm} (27)
\[ \cos u - \cos v = -2 \sin \frac{1}{2} (u + v) \sin \frac{1}{2} (u - v) . \]  \hspace{1cm} (28)

Dividing (25) by (26),

\[ \frac{\sin u + \sin v}{\sin u - \sin v} = \frac{\tan \frac{1}{2} (u + v)}{\tan \frac{1}{2} (u - v)} . \]  \hspace{1cm} (29)

EXERCISES

39. Express in terms of functions of \( x \), by means of the formulas of this chapter,
(1.) Tan (180° — x); tan (180° + x).

(2.) The functions of (x — 180°).

(3.) Sin (x — 90°) and cos (x — 90°).

(4.) Sin (x — 270°), and cos (x — 270°).

(5.) The sine and cosine of (45° — x); of (45° + x).

(6.) Given tan 45° = 1, tan 30° = \( \frac{1}{\sqrt{3}} \); find tan 75°; tan 15°.

(7.) Prove \( \cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x} \).

\( \text{Hint.} \) — Divide formula (13) by formula (11).

(8.) Prove \( \cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x} \).

(9.) Prove cos (30° + y) = cos (30° — y) = sin y.

(10.) Prove sin 3x = 3 sin x — 4 sin^3 x.

\( \text{Hint.} \) — Sin 3x = sin (x + 2x).

(11.) Prove cos 3x = 4 cos^3 x — 3 cos x.

(12.) If x and y are acute and tan x = \( \frac{1}{3} \), tan y = \( \frac{1}{3} \), prove that (x + y) = 45°.

(13.) Prove that tan (x + 45°) = \( \frac{1 + \tan x}{1 - \tan x} \).

(14.) Given sin y = \( \frac{1}{3} \) and y acute; find sin \( \frac{1}{2} y \), cos \( \frac{1}{2} y \), and tan \( \frac{1}{2} y \).

(15.) Given cos x = \( \frac{1}{2} \) and x in quadrant II; find sin 2x and cos 2x.

(16.) Given cos 45° = \( \frac{1}{2} \sqrt{2} \); find the functions of 22\( \frac{1}{2} \)°.

(17.) Given tan x = 2 and x acute; find tan \( \frac{1}{2} x \).

(18.) Given cos 30° = \( \frac{1}{2} \sqrt{3} \); find the functions of 15°.

(19.) Given cos 90° = 0; find the functions of 45°.

(20.) Find sin 5x in terms of sin x.

(21.) Find cos 5x in terms of cos x.

(22.) Prove sin (x + y + z) = sin x sin y cos z + cos x sin y cos z + cos x cos y sin z — sin x sin y sin z.

\( \text{Hint.} \) — Sin (x + y + z) = sin (x + y) cos z + cos (x + y) sin z.

(23.) Given tan 2x = 3 tan x; find x.

(24.) Prove sin 32° + sin 28° = cos 2°.

(25.) Prove tan x + cot x = 2 cosec 2x.

(26.) Prove (sin \( \frac{1}{2} x \) + cos \( \frac{1}{2} x \))^2 = 1 + sin x.

(27.) Prove (sin \( \frac{1}{2} x \) — cos \( \frac{1}{2} x \))^2 = sin x.
(28.) Prove \( \cos 2x = \cos^2 x - \sin^2 x \).

(29.) Prove \( \tan (45^\circ + x) + \tan (45^\circ - x) = 2 \sec 2x \).

(30.) Prove \( \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} \).

(31.) Prove \( \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} \).

(32.) Prove \( \frac{1 + \sin 2x}{1 - \sin 2x} = (\frac{\tan x + 1}{\tan x - 1})^2 \).

(33.) Prove \( \tan \frac{1}{2} x = \frac{\sin x}{1 + \cos x} \).

(34.) Prove \( \cot \frac{1}{2} x = \frac{\sin x}{1 - \cos x} \).

(35.) Express as a product \( \frac{\cos x - \cos y}{\cos x + \cos y} \).

\textbf{Hint.} \quad \frac{\cos x - \cos y}{\cos x + \cos y} = \frac{-2 \sin \frac{1}{2} (x+y) \sin \frac{1}{2} (x-y)}{2 \cos \frac{1}{2} (x+y) \cos \frac{1}{2} (x-y)} = -\tan \frac{1}{2} (x+y) \tan \frac{1}{2} (x-y) \).

(36.) Express as a product \( \frac{\tan x + \tan y}{\cot x + \cot y} \).

(37.) Prove \( 1 - \tan x \tan y = \frac{\cos(x + y)}{\cos x \cos y} \).

**THE INVERSE TRIGONOMETRIC FUNCTIONS**

40. **Def.**—The expressions \( \sin^{-1} a \), \( \cos^{-1} a \), \( \tan^{-1} a \), etc., denote respectively an angle whose sine is \( a \), an angle whose cosine is \( a \), an angle whose tangent is \( a \), etc. They are called the **inverse sine** of \( a \), the **inverse cosine** of \( a \), the **inverse tangent** of \( a \), etc., and are the **inverse trigonometric functions**.

\( \sin^{-1} a \) is an angle whose sine is equal to \( a \), and hence denotes, not a single definite angle, but each and every angle whose sine is \( a \).

* Since quantities cannot be added or subtracted by the ordinary operations with logarithms, an expression must be reduced to a form in which no addition or subtraction is required, to be convenient for logarithmic computation.
Thus, if \( \sin x = \frac{1}{2}, x = 30^\circ, 150^\circ, (30^\circ + 360^\circ), \) etc., and \( \sin^{-1} \frac{1}{2} = 30^\circ, 150^\circ, (30^\circ + 360^\circ), \) etc.

**Remark.**—The sine or cosine of an angle cannot be less than \(-1\) or greater than \(1\); hence \( \sin^{-1} a \) and \( \cos^{-1} a \) have no meaning unless \( a \) is between \(-1\) and \(1\). In a similar manner we see that \( \sec^{-1} a \) and \( \csc^{-1} a \) have no meaning if \( a \) is between \(-1\) and \(1\).

**EXERCISES**

41. (1.) Find the following angles in degrees:
   - \( \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) \)
   - \( \tan^{-1}(-1) \)
   - \( \sin^{-1}(-\frac{1}{2}) \)
   - \( \cos^{-1} \frac{1}{2} \)
   - \( \cos^{-1} 1 \)

(2.) If \( x = \cot^{-1} \frac{1}{2} \), find \( \tan x \).
(3.) If \( x = \sin^{-1} \frac{1}{2} \), find \( \cos x \) and \( \tan x \).
(4.) Find \( \sin (\tan^{-1} \frac{\sqrt{3}}{2}) \).
(5.) Find \( \sin (\cos^{-1} \frac{1}{2}) \).
(6.) Find \( \cot (\tan^{-1} \frac{1}{4}) \).
(7.) Given \( \sin^{-1} a = 2 \cos^{-1} a \), and both angles acute; find \( a \).
(8.) Given \( \sin^{-1} a = \cos^{-1} a \); find the values of \( \sin^{-1} a \) less than \(360^\circ\).
(9.) Given \( \tan^{-1} 1 = \frac{1}{2} \tan^{-1} 2 \), and both angles less than \(360^\circ\); find the angles.
(10.) Given \( \sin^{-1} a = \cos^{-1} a \) and \( \sin^{-1} a + \cos^{-1} a = 450^\circ \); find \( \sin^{-1} a \).
(11.) Prove \( \sin (\cos^{-1} a) = \pm \sqrt{1 - a^2} \).

**Hint.**—Let \( x = \cos^{-1} a \); then \( a = \cos x \),
   \[ \sin x = \pm \sqrt{1 - \cos^2 x} = \pm \sqrt{1 - a^2}. \]

(12.) Prove \( \tan (\tan^{-1} a + \tan^{-1} b) = \frac{a + b}{1 - ab} \).
(13.) Prove \( \tan (\tan^{-1} a - \tan^{-1} b) = \frac{a - b}{1 + ab} \).
(14.) Prove \( \cos (2 \cos^{-1} a) = 2a^2 - 1 \).
(15.) Prove \( \sin (2 \cos^{-1} a) = \pm 2a \sqrt{1 - a^2} \).
(16.) Prove \( \tan (2 \tan^{-1} a) = \frac{2a}{1 - a^2} \).
(17.) Prove \( \cos (2 \tan^{-1} a) = \frac{1 - a^2}{1 + a^2} \).
(18.) Prove \( \sin (\sin^{-1} a + \cos^{-1} b) = ab \pm \sqrt{(1 - a^2)(1 - b^2)} \).
CHAPTER IV
THE OBLIQUE TRIANGLE
DERIVATION OF FORMULAS

42. The formulas derived in this and the succeeding articles reduce the solution of the oblique triangle to its simplest form.

Draw the perpendicular $CD$. Let $CD = h$,

Then \[ \frac{h}{b} = \sin A; \]

(In Fig. 2 \[ \frac{h}{b} = \sin (180^\circ - A) = \sin A \])

and \[ \frac{h}{a} = \sin B. \]

(In Fig. 3 \[ \frac{h}{a} = \sin (180^\circ - B) = \sin B. \])

By division we obtain,

\[ \frac{a}{b} = \frac{\sin A}{\sin B}. \] \quad (32)

Remark.—This formula expresses the fact that the ratio of two sides of an oblique triangle is equal to the ratio of the sines of the angles opposite, and does not in any respect depend upon which side has been taken as the base. Hence if the letters are advanced one step, as shown in the figure, we obtain, as another form of the same formula,
Repeating the process, we obtain
\[
\frac{c}{\sin C} = \frac{a}{\sin A}.
\]

The same procedure may be applied to all the formulas for the solution of oblique triangles. Henceforth only one expression of each formula will be given.

Formula (32) is used for the solution of triangles in which a side and two angles, or two sides and an angle, opposite one of them are given.

43. We obtain from formula (32) by division and composition,
\[
\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.
\]

By formula (29), denoting the angles by \(A\) and \(B\), instead of \(u\) and \(v\),
\[
\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.
\]

Therefore,
\[
\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.
\] (33)

This formula is used for the solution of triangles in which two sides and the included angle are given.

44. Whether \(A\) is acute or obtuse, we have

(If \(A\) is acute (Fig. 1), \(AD = b \cos A, DB = AB - AD = c - b \cos A, CD = b \sin A\). If \(A\) is obtuse (Fig. 2), \(AD = b \cos (180° - A) = -b \cos A, DB = AB + AD = c - b \cos A, CD = b \sin (180° - A) = b \sin A\).)
THE OBLIQUE TRIANGLE

\[ a^2 = (c - b \cos A)^2 + (b \sin A)^2, \]

\[ = c^2 - 2bc \cos A + b^2 (\cos^2 A + \sin^2 A). \]

Therefore, \( a^2 = b^2 + c^2 - 2bc \cos A. \) \hspace{1cm} (34)

This formula is used in deriving formula (37).

It is also used in the solution without logarithms of triangles of which two sides and the included angle or three sides are given.

45. From formula (34), \( \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \)

From formula (22), § 37,

\[ 2 \sin^2 \frac{1}{2} A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}. \]

Hence \( 2 \sin^2 \frac{1}{2} A = \frac{2bc + a^2 - b^2 - c^2}{2bc}, \)

\[ = \frac{a^2 - (b - c)^2}{2bc}, \]

\[ = \frac{(a - b + c)(a + b - c)}{2bc}. \]

Let \( s = \frac{a + b + c}{2}, \) then \( (a - b + c) = 2(s - b), \) and \( (a + b - c) = 2(s - c). \)

Substituting, \( 2 \sin^2 \frac{1}{2} A = \frac{2(s - b)(s - c)}{bc}. \)

Hence \( \sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}. \) \hspace{1cm} (35)

From formula (23), § 37,

\[ 2 \cos^2 \frac{1}{2} A = 1 + \cos A = \frac{2bc + b^2 + c^2 - a^2}{2bc}, \]

\[ = \frac{2s(s - a)}{bc}. \]

*In extracting the root the plus sign is chosen because it is known that \( \sin \frac{1}{2} A \) is positive.*
Hence  \[ \cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}. \]  (36)

Dividing (35) by (36), we obtain

\[ \tan \frac{1}{2} A = \frac{\sqrt{(s-b)(s-c)}}{s(s-a)}, \]  (37)

\[ = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s(s-a)^2}, \]

\[ = \frac{1}{s-a} \sqrt{(s-a)(s-b)(s-c)} \cdot \frac{s}{s}. \]

Let  \[ K = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s}. \]

\[ \tan \frac{1}{2} A = \frac{K}{s-a}. \]  (38)

Formulas (37) and (38) are used to find the angles of a triangle when the three sides are given.

FORMULAS FOR THE AREA OF A TRIANGLE

46. Denote the area by \( S \).

\[ (\text{in Fig. 1, } CD = a \sin B; \text{ in Fig. 2, } CD = a \sin (180^\circ - B) = a \sin B.) \]

In Figs. 1 and 2,  \[ S = \frac{1}{2} c \cdot CD. \]

Hence  \[ S = \frac{1}{2} ac \sin B. \]  (39)

From formula (17),

\[ \sin B = 2 \sin \frac{1}{2} B \cos \frac{1}{2} B. \]
Substituting for $\sin \frac{1}{4} B$ and $\cos \frac{1}{4} B$ the values found in formulas (35) and (36), we obtain

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore,

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$  \hspace{1cm} (40)

This formula may also be written,

$$S = sK.$$  \hspace{1cm} (41)

Formula (39) is used to find the area of a triangle when two sides and the included angle are known; formula (40) or formula (41), when the three sides are known.

THE AMBIGUOUS CASE

47. The given parts are two sides, and an acute angle opposite one of them.

Let these parts be denoted by $a$, $b$, $A$.

If $a$ is less than $b$ and greater than the perpendicular $CD$ (Fig. 1), there are the two triangles $ACB$ and $ACB'$, which contain the given parts, or, in other words, there are two solutions.

If $a$ is greater than $b$ (Fig. 2), there is one solution.

If $a$ is equal to the perpendicular $CD$, there is one solution, the right triangle $ACD$. 
If the given value of $a$ is less than $CD$, evidently there can be no triangle containing the given parts.

Since $CD=b \sin A$, there is no solution when $a < b \sin A$; there is one solution, the right triangle $ACD$ when $a=b \sin A$; there are two solutions when $a < b$ and $> b \sin A$.

48. Case I.—Given a side and two angles.

**EXAMPLE**

Given $a = 36.738, A = 36^\circ 55' 54''$, $B = 72^\circ 5' 56''$, $C = 180^\circ - (A + B) = 180^\circ - 109^\circ 1' 50'' = 70^\circ 58' 10''$.

To find $b$.

\[
\frac{b}{a} = \frac{\sin B}{\sin A}
\]

\[
\log a = 1.56512
\]
\[
\log \sin B = 0.97845 - 10
\]
\[
colog \sin A = 0.22123
\]
\[
\log b = 1.76480
\]
\[
b = 58.184
\]

To find $c$.

\[
\frac{c}{a} = \frac{\sin C}{\sin A}
\]

\[
\log a = 1.56512
\]
\[
\log \sin C = 0.97559 - 10
\]
\[
colog \sin A = 0.22123
\]
\[
\log c = 1.76194
\]
\[
c = 57.80
\]

**Check.**

Determine $b$ from $c$, $C$, and $B$ by the formula

\[
\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}
\]

This check is long, but is quite certain to reveal an error. A check which is shorter, but less sure, is

\[
\frac{b}{c} = \frac{\sin B}{\sin C}
\]

Solve the following triangles:

1. Given $a = 567.25, A = 11^\circ 15', B = 47^\circ 12'$.
2. Given $a = 783.29, A = 81^\circ 52', B = 42^\circ 27'$.
3. Given $c = 1125.2, A = 79^\circ 15', B = 55^\circ 11'$.
4. Given $b = 15.346, B = 15^\circ 51', C = 58^\circ 10'$.
5. Given $a = 5301.5, A = 69^\circ 44', C = 41^\circ 18'$.
6. Given $b = 1002.1, A = 48^\circ 59', C = 76^\circ 3'$.

49. Case II.—Given two sides of a triangle and the angle opposite one of them.
EXAMPLE

Given \( a = 23.203 \), \( b = 35.121 \), \( A = 36^\circ 8' 10'' \).

To find \( B \) and \( B' \).

\[
\begin{align*}
\sin B &= \frac{b}{\sin A} = \frac{35.121}{23.203} \\
\log b &= 1.54556 \\
\log \sin A &= 9.77064 - 10 \\
\text{colog} a &= 8.63445 - 10 \\
\log \sin B &= 9.95065 - 10 \\
B &= 63^\circ 12' \\
B' &= 180^\circ - B = 116^\circ 48'
\end{align*}
\]

To find \( C \) and \( C' \).

\[
\begin{align*}
C &= 180^\circ - (A + B) = 80^\circ 39' 50'' \\
C' &= 180^\circ - (A + B') = 27^\circ 3' 50''
\end{align*}
\]

Check.

Determine \( b \) from \( c, C, \) and \( B \) by the formula

\[
\frac{b-a}{b+a} = \tan \frac{1}{2} (B-A)
\]

This check is long, but is quite certain to reveal an error. A check which is shorter, but less sure, is

\[
\frac{b}{c} = \frac{\sin B}{\sin C}
\]

(1.) How many solutions are there in each of the following?

(1.) \( A = 30^\circ, a = 15, b = 20 \);
(2.) \( A = 30^\circ, a = 10, b = 20 \);
(3.) \( B = 30^\circ, a = 8, b = 20 \);
(4.) \( B = 37^\circ 23', a = 9.1, b = 7.5 \).
Solve the following triangles, finding all possible solutions:

(2.) Given \( A = 147^\circ 12' \), \( a = 0.63735 \), \( b = 0.34312 \).

(3.) Given \( A = 24^\circ 31' \), \( a = 1.7424 \), \( b = 0.96245 \).

(4.) Given \( A = 21^\circ 21' \), \( a = 45.693 \), \( b = 56.723 \).

(5.) Given \( A = 61^\circ 16' \), \( a = 9.5124 \), \( b = 12.752 \).

(6.) Given \( C = 22^\circ 32' \), \( a = 0.78727 \), \( c = 0.47311 \).

\[50. \text{ Case III.--Given two sides and the included angle.}\]

**EXAMPLE**

Given \( a = 41.003 \), \( b = 48.718 \), \( C = 68^\circ 33' 58'' \); find the remaining parts and the area.

\[To \ find \ A \ and \ B.\]

\[
\begin{align*}
\frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)} &= \frac{b-a}{b+a} \\
&= \frac{7.715}{89.721} \\
&= 0.08734 \\
\log (b-a) &= 0.88734 \\
colog (b+a) &= 8.094710 - \infty \\
\log \tan \frac{1}{2}(B+A) &= 0.16639 \\
\log \tan \frac{1}{2}(B-A) &= 9.10083 - \infty \\
\frac{1}{2}(B-A) &= 7^\circ 11' 20'' \\
\frac{1}{2}(B+A) &= 55^\circ 43' 1'' \\
B &= 62^\circ 54' 21'' \\
A &= 48^\circ 31' 41''
\end{align*}
\]

\[To \ find \ c.\]

\[
\frac{c}{a} = \sin C \\
&= 0.50938 \\
\log a &= 1.61281 \\
\log \sin C &= 9.96888 - \infty \\
colog A &= 0.12535 \\
\log c &= 1.70704 \\
c &= 50.938 \\
To \ find \ the \ area.\]

\[
S = \frac{1}{2} ab \sin C \\
&= 929.72 \\
\frac{1}{2} ab &= 9.69897 - \infty \\
\log a &= 1.61281 \\
\log b &= 1.68769 \\
\log \sin C &= 9.96888 - \infty \\
\log S &= 2.96835 \\
S &= 929.72
\]

**Check.**

\[
\begin{align*}
\sin C &= \frac{c}{a} \\
\sin B &= \frac{b}{a} \\
\log \sin B &= 9.94951 - \infty \\
&= 1.70704 \\
colog b &= 8.31231 - \infty \\
\log \sin C &= 9.06886 - \infty
\end{align*}
\]
THE OBLIQUE TRIANGLE

Solve the following triangles, and also find their areas:

(1.) Given $A = 41^\circ 15', b=0.14726, c=0.10971$.

(2.) Given $C = 58^\circ 47', b=11.726, a=16.147$.

(3.) Given $B = 49^\circ 50', a=103.74, c=99.975$.

(4.) Given $A = 33^\circ 31', b=0.32041, c=0.9203$.

(5.) Given $C=128^\circ 7', b=17.738, a=60.571$.

51. Case IV.—Given the three sides.

EXAMPLE

Given $a = 32.456, b = 41.724, c = 53.987$; find the angles and area.

\[
\begin{align*}
K &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \\
\log (s-a) &= 1.50007 \\
\log (s-b) &= 1.34947 \\
\log (s-c) &= 1.00419 \\
\text{colog } s &= 8.19325 - 10 \\
\log K &= 1.02349
\end{align*}
\]

To find $B$.

\[
\begin{align*}
\tan \frac{1}{2} B &= \frac{K}{s-b} \\
\log K &= 1.02349 \\
\log (s-b) &= 1.34947 \\
\log \tan \frac{1}{2} B &= 9.67402 - 10 \\
\frac{1}{2} B &= 25^\circ 16' 16'' \\
B &= 50^\circ 32' 32''
\end{align*}
\]

To find $C$.*

\[
\begin{align*}
\tan \frac{1}{2} C &= \frac{K}{s-c} \\
\log K &= 1.02349 \\
\log (s-c) &= 1.00419 \\
\log \tan \frac{1}{2} C &= 0.01930 \\
\frac{1}{2} C &= 46^\circ 16' 22'' \\
C &= 92^\circ 32' 44''
\end{align*}
\]

Check.

\[A + B + C = 180^\circ 0' 2''\]

Find the angles and areas of the following triangles:

(1.) Given $a = 38.516, b=44.873, c=14.517$.

(2.) Given $a = 2.1158, b=3.5854, c=3.5679$.

* $C$ could be found from $(A + B) = (180^\circ - C)$, but for the sake of the check it is worked out independently.
(3.) Given $a = 82.818, b = 99.871, c = 36.363$.

(4.) Given $a = 36.789, b = 51.1.7, c = 33.328$.

(5.) Given $a = 113.08, b = 113.17, c = 114.29$.

(6.) Given $a = .9763, b = 1.2489, c = 1.6543$.

**EXERCISES**

52. (1.) A tree, $A$, is observed from two points, $B$ and $C$, 1863 ft. apart on a straight road. The angle $BCA$ is $36^\circ 43'$, and the angle $CBA$ is $57^\circ 21'$.

Find the distance of the tree from the nearer point.

(2.) Two houses, $A$ and $B$, are 3876 yards apart. How far is a third house, $C$, from $A$, if the angles $ABC$ and $BAC$ are $49^\circ 17'$ and $58^\circ 18'$ respectively?

(3.) A triangular lot has one side 285.4 ft. long. The angles adjacent to this side are $41^\circ 22'$ and $31^\circ 19'$. Find the length of a fence around it, and its area.

(4.) The two diagonals of a parallelogram are 8 and 10, and the angle between them is $53^\circ 8'$; find the sides of the parallelogram.

(5.) Two mountains, $A$ and $B$, are 9 and 13 miles from a town, $C$; the angle $ACB$ is $71^\circ 36' 37''$. Find the distance between the mountains.

(6.) Two buoys are 2789 ft. apart, and a boat is 4325 ft. from the nearer buoy. The angle between the lines from the buoys to the boat is $16^\circ 13'$. How far is the boat from the farther buoy? Are there two solutions?

(7.) Given $a = 64.256, c = 19.278, C = 16^\circ 19' 11''$; find the difference in the areas of the two triangles which have these parts.

(8.) A prop 13 ft. long is placed 6 ft. from the base of an embankment, and reaches 8 ft. up its face; find the slope of the embankment.

(9.) The bounding lines of a township form a triangle of which the sides are 8.943 miles, 7.2415 miles, and 10.817 miles; find the area of the township.

(10.) Prove that the diameter of a circle circumscribed about a triangle is equal to any side of the triangle divided by the sine of the angle opposite.
Hint.—By Geometry, \( \angle AOB = 2C \).

Draw \( OD \) perpendicular to \( AB \).

Angle \( DOB = \frac{1}{2} AOB = C \).

\[ DB = r \sin DOB = r \sin C. \]

Hence

\[ c = 2r \sin C, \]

or

\[ 2r = \frac{c}{\sin C}. \]

(11.) The distances \( AB, BC, \) and \( AC \), between three cities, \( A, B, \) and \( C \), are 12 miles, 14 miles, and 17 miles respectively. Straight railroads run from \( A \) to \( B \) and \( C \). What angle do they make?

(12.) A balloon is directly over a straight road, and between two points on the road from which it is observed. The points are 15847 ft. apart, and the angles of elevation are found to be 49° 12' and 53° 29' respectively. Find the distance of the balloon from each of the points.

(13.) To find the distance from a point \( A \) to a point \( B \) on the opposite side of a river, a line, \( AC \), and the angles \( CAB \) and \( ACB \) were measured and found to be 315.32 ft., 58° 43', and 57° 13' respectively. Find the distance \( AB \).

(14.) A building 50 ft. high is situated on the slope of a hill. From a point 200 ft. away the building subtends an angle of 12° 13'. Find the distance from this point to the top of the building.

(15.) Prove that the area of a quadrilateral is equal to one-half the product of the diagonals by the sine of the angle between them.

(16.) From points \( A \) and \( B \), at the bow and stern of a ship respectively, the foremast, \( C \), of another ship is observed. The points \( A \) and \( B \) are 300 ft. apart; the angles \( ABC \) and \( BAC \) are found to be
65° 31' and 110° 46' respectively. What is the distance between the points \( A \) and \( C \) of the two ships?

(17.) Two steamers leave the same port at the same time; one sails, directly northwest, 12 miles an hour; the other 17 miles an hour, in a direction 67° south of west. How far apart will they be at the end of three hours?

(18.) Two stakes, \( A \) and \( B \), are on opposite sides of a stream; a third stake, \( C \), is set 62 ft. from \( A \); the angles \( ACB \) and \( CAB \) are found to be 50° 3' 5" and 61° 18' 20" respectively. How long is a rope connecting \( A \) and \( B \)?

(19.) To find the distance between two inaccessible mountain-tops, \( A \) and \( B \), of practically the same height, two points, \( C \) and \( D \), are taken one mile apart. The angle \( CDA \) is found to be 88° 34', the angle \( DCA \) is 63° 8', the angle \( CD B \) is 64° 27', the angle \( DCB \) is 87° 9'. What is the distance?

(20.) Two islands, \( B \) and \( C \), are distant 5 and 3 miles respectively from a light-house, \( A \), and the angle \( BAC \) is 33° 11'; find the distance between the islands.

(21.) Two points, \( A \) and \( B \), are visible from a third point \( C \), but not from each other; the distances \( AC \), \( BC \), and the angle \( ACB \) were measured, and found to be 1321 ft., 1287 ft., and 61° 22' respectively. Find the distance \( AB \).

(22.) Of three mountains, \( A \), \( B \), and \( C \), \( B \) is directly north of \( C \) 5 miles, \( A \) is 8 miles from \( C \) and 11 from \( B \). How far is \( A \) south of \( B \)?

(23.) From a position 215.75 ft. from one end of a building and 198.25 ft. from the other end, the building subtends an angle of 53° 37' 28"; find its length.

(24.) If the sides of a triangle are 372.15, 427.82, and 404.17; find the cosine of the smallest angle.

(25.) From a point 3 miles from one end of an island and 7 miles from the other end, the island subtends an angle of 33° 55' 15"; find the length of the island.

(26.) A point is 13581 in. from one end of a wall 12342 in. long, and 10025 in. from the other end. What angle does the wall subtend at this point?

(27.) A straight road ascends a hill a distance of 213.2 ft., and is in-
THE OBLIQUE TRIANGLE

clined 12° 2' to the horizontal; a tree at the bottom of the hill subtends at the top an angle of 10° 5' 16". Find the height of the tree.

(28.) Two straight roads cross at an angle of 37° 50' at the point A; 3 miles distant on one road is the town B, and 5 miles distant on the other is the town C. How far are B and C apart?

(29.) Two stations, A and B, on opposite sides of a mountain, are both visible from a third station, C; \( AC = 11.5 \) miles, \( BC = 9.4 \) miles, and the angle \( ACB = 59°\ 31' \). Find the distance from A to B.

(30.) To obtain the distance of a battery, A, from a point, B, of the enemy's lines, a point, C, 372.7 yards distant from A is taken; the angles \( ACB \) and \( CAB \) are measured and found to be 79° 53' and 74° 35' respectively. What is the distance \( AB \)?

(31.) A town, B, is 14 miles due west of another town, A. A third town, C, is 19 miles from A and 17 miles from B. How far is C west of A?

(32.) Two towns, A and B, are on opposite sides of a lake. A is 18 miles from a third town, C, and B is 13 miles from C; the angle \( ACB \) is 13° 17'. Find the distance between the towns A and B.

(33.) At a point in a level plane the angle of elevation of the top of a hill is 39° 51', and at a point in the same direct line from the hill, but 217.2 feet farther away, the angle of elevation is 26° 53'. Find the height of the hill above the plane.

(34.) It is required to find the distance between two inaccessible points, A and B. Two stations, C and D, 2547 ft. apart, are chosen and the angles are measured; they are \( ACB = 27°\ 21' \), \( BCD = 33°\ 14' \), \( BDA = 18°\ 17' \), and \( ADC = 51°\ 23' \). Find the distance from A to B.

(35.) Two trains leave the same station at the same time on straight tracks inclined to each other 21° 12'. If their average speeds are 40 and 50 miles an hour, how far apart will they be at the end of the first fifteen minutes?

(36.) A ship, A, is seen from a light-house, B; to determine its distance a point, C, 300 ft. from the light-house is taken and the angles \( BCA \) and \( CBA \) measured. If \( BCA = 108°\ 34' \) and \( CBA = 65°\ 27' \), what is the distance of the ship from the light-house?
(37.) Prove that the radius of the inscribed circle of a triangle is equal to \( a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A. \)

_Hint._—Draw \( OB, OC, \) and the perpendicular \( OD. \)

\( OB \) and \( OC \) bisect the angles \( B \) and \( C \) respectively, and \( OD = r. \)

\[
a = BD + DC = r (\cot \frac{1}{2} B + \cot \frac{1}{2} C).
\]

\[
\cot \frac{1}{2} B + \cot \frac{1}{2} C = \frac{\sin \frac{1}{2} C \cos \frac{1}{2} B + \cos \frac{1}{2} C \sin \frac{1}{2} B}{\sin \frac{1}{2} B \sin \frac{1}{2} C} = \frac{\sin \frac{1}{2} (B + C)}{\sin \frac{1}{2} B \sin \frac{1}{2} C} = \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} B \sin \frac{1}{2} C}.
\]

Hence

\[
r = a \frac{\sin \frac{1}{2} B \sin \frac{1}{2} C}{\cos \frac{1}{2} A} = a \sin \frac{1}{2} B \sin \frac{1}{2} C \sec \frac{1}{2} A.
\]
CHAPTER V
CIRCULAR MEASURE—GRAPHICAL REPRESENTATION

CIRCULAR MEASURE

53. The length of the semicircumference of a circle is \( \pi R (\pi = 3.14159 + ) \); the angle the semicircumference subtends at the centre of the circle is \( 180^\circ \). Hence an arc whose length is equal to the radius will subtend the angle \( \frac{180^\circ}{\pi} \); this angle is the unit angle of circular measure, and is called a radian.

If the radius of the circle is unity, an arc of unit length subtends a radian; hence in the unit circle the length of an arc represents the circular measure of the angle it subtends.

Thus, if the length of an arc is \( \frac{\pi}{2} \), it subtends the angle \( \frac{\pi}{2} \) radians.

Since one radian = \( \frac{180^\circ}{\pi} \), we have

\[
90^\circ = \frac{\pi}{2} \text{ radians},
\]

\[
180^\circ = \pi \text{ radians},
\]
\[ 270° = \frac{3\pi}{2} \text{ radians,} \]
\[ 360° = 2\pi \text{ radians, etc.} \]

The value of a radian in degrees and of a degree in radians are:

1 radian = 57.29578°,
\[ = 57° 17' 45". \]
1° = 0.0174533 radian.

In the use of the circular measure it is customary to omit the word radian, thus we write \( \frac{\pi}{2}, \pi, \) etc., denoting \( \frac{\pi}{2} \) radians, \( \pi \) radians, etc. On the other hand, the symbols ° ′ ″ are always printed if an angle is measured in degrees, minutes, and seconds; hence there is no confusion between the systems.

**EXERCISES**

(1.) Express in circular measure 30°, 45°, 60°, 120°, 135°, 720°, 990°.
(Take \( \pi = 3.1416. \))

(2.) Express in degrees, minutes, and seconds the angles \( \frac{\pi}{8}, \frac{\pi}{10}, \frac{1}{2}, \frac{7}{4}. \)

(3.) What is the circular measure of the angle subtended by an arc of length 2.7 in., if the radius of the circle is 2 in.? if the radius is 5 in.?  

54. The following important relations exist between the circular measure \( x \) of an angle and the sine and tangent of the angle.

(1.) *If \( x \) is less than \( \frac{\pi}{2}, \sin x < x < \tan x. \)

\[ \text{Draw a circle of unit radius.} \]

By Geometry, \( SP < \text{arc } AP < AT. \)

Hence \( \sin x < x < \tan x. \)
(2.) As \( x \) approaches the limit 0, \( \frac{\sin x}{x} \) and \( \frac{\tan x}{x} \) approach the limit 1.

Dividing \( \sin x < x < \tan x \) by \( \sin x \), we obtain

\[
1 < \frac{x}{\sin x} < \frac{1}{\cos x}.
\]

Inverting,

\[
1 > \frac{\sin x}{x} > \frac{\cos x}{1}.
\]

As \( x \) approaches the limit 0, \( \cos x \) approaches the length of the radius, that is, 1, as a limit.

Therefore, \( \frac{\sin x}{x} \) approaches the limit 1.

Dividing \( 1 > \frac{\sin x}{x} > \cos x \) by \( \cos x \), we obtain

\[
\frac{1}{\cos x} > \frac{\tan x}{x} > 1.
\]

As \( x \) approaches the limit 0, \( \cos x \) approaches the limit 1; hence \( \frac{1}{\cos x} \) approaches the limit 1.

Therefore, \( \frac{\tan x}{x} \) approaches the limit 1.

**PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS**

55. The sine of an angle \( x \) is the same as the sine of \( (x + 360^\circ) \), \( (x + 720^\circ) \), etc.—that is, of \( (x + 2n\pi) \), where \( n \) is any integer.

The sine is therefore said to be a periodic* function, having the period \( 360^\circ \), or \( 2\pi \).

The same is true of the cosine, secant, and cosecant.

* If a function, denoted by \( f(x) \), of a variable \( x \), is such that \( f(x + k) = f(x) \) for every value of \( x \), \( k \) being a constant, the function \( f(x) \) is periodic; if \( k \) is the least constant which possesses this property, \( k \) is the period of \( f(x) \).
The tangent of an angle $x$ is the same as the tangent of $(x + 180^\circ), (x + 360^\circ), \text{etc.} -$ that is, of $(x + n\pi)$, where $n$ is any integer. The tangent is therefore a periodic function, having the period $180^\circ$, or $\pi$. The same is true of the cotangent.

**GRAPHICAL REPRESENTATION**

56. On the line $OX$ lay off the distance $OA(=x)$ to represent the circular measure of the angle $x$. At the point $A$ erect a perpendicular equal to $\sin x$. If perpendiculars are thus erected for each value of $x$, the curve passing through their extremities is called the sine curve. If $\sin x$ is negative, the perpendicular is drawn downward.

In a similar manner the cosine, tangent, cotangent, secant, and cosecant curves can be constructed.
Tangent Curve

Cotangent Curve
If the distances on $OX$ are measured from $O'$ instead of $O$, we obtain from the secant curve the cosecant curve.

In the construction of the inverse curves the number is represented by the distance to the right or left from $O$; the circular measure of the angle by the length of the perpendicular erected.

All of the preceding curves, except the tangent and cotangent curves, have a period of $2\pi$ along the line $OX$; that is, the curve extended in either direction is of the same form in each case between $2\pi$ and $4\pi$, $4\pi$ and $6\pi$, $-2\pi$ and $0$, etc., as between $0$ and $2\pi$, while the corresponding inverse curves repeat along the vertical line in the same period. The period of the tangent and cotangent curves is $\pi$. 
GRAPHICAL REPRESENTATION

**Inverse Sine Curve**

**Inverse Cosine Curve**

**Inverse Tangent Curve**
Inverse secant
CHAPTER VI
COMPUTATION OF LOGARITHMS AND OF THE TRIGONOMETRIC FUNCTIONS−DE MOIVRE'S THEOREM—HYPERBOLIC FUNCTIONS

57. A convenient method of calculating logarithms and the trigonometric functions is to use infinite series. In works on the Differential Calculus it is shown that

\[
\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \tag{1}
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots * \tag{2}
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \tag{3}
\]

Another development which we shall use later is

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{4}
\]

where \(e = 2.7182818 \ldots\) is the base of the Naperian system of logarithms.

58. The series (1) converges only for values of \(x\) which satisfy the inequality \(-1 < x \leq 1\). The series (2), (3), and (4) converge for all finite values of \(x\).

It is to be noted that the logarithm in (1) is the Naperian, and the angle \(x\) in (2) and (3) is expressed in circular measure.

* \(3!\) denotes \(1 \times 2 \times 3\); \(4!\) denotes \(1 \times 2 \times 3 \times 4\), etc.
COMPUTATION OF LOGARITHMS

59. We first recall from Algebra the definition and some of the principal theorems of logarithms.

The logarithm to the base $a$ of the number $m$ is the number $x$ which satisfies the equation,

$$a^x = m.$$  

This is written $x = \log_a m$.

The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Thus

$$\log_a {mn} = \log_a m + \log_a n.$$  

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

Thus

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$  

The logarithm of the power of a number is equal to the logarithm of the number multiplied by the exponent.

Thus

$$\log_a {m^p} = p \log_a m.$$  

To obtain the logarithm of a number to any base $a$ from its Napierian logarithm, we have

$$\log_a m = \frac{\log_e m}{\log_e a} = M_a \log_e m,$$  

where $M_a = \frac{1}{\log_e a}$; $M_a$ is called the modulus of the system.

60. We proceed now to the computation of logarithms. The series (1) enables us to compute directly the Napierian logarithms of positive numbers not greater than 2.

Example.—To compute $\log_e \frac{3}{2}$ to five places of decimals.

Substitute $\frac{1}{2}$ for $x$ in (1):

$$\log_e \frac{3}{2} = \log_e \left(1 + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} - \frac{1}{4} \cdot \frac{1}{2^4} + \ldots$$

If the result is to be correct to five places of decimals, we must take enough terms so that the remainder shall not affect the fifth decimal place. Now we
know by Algebra that in a series of which the terms are each less in numerical value than the preceding, and are also alternately positive and negative, the remainder is less in numerical value than its first term. Hence we need to take enough terms to know that the first term neglected would not affect the fifth place.

### Positive terms

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} )</td>
<td>0.500000</td>
</tr>
<tr>
<td>( \frac{1}{3} \cdot \frac{1}{2^3} )</td>
<td>0.041667</td>
</tr>
<tr>
<td>( \frac{1}{5} \cdot \frac{1}{2^5} )</td>
<td>0.006250</td>
</tr>
<tr>
<td>( \frac{1}{7} \cdot \frac{1}{2^7} )</td>
<td>0.0011161</td>
</tr>
<tr>
<td>( \frac{1}{9} \cdot \frac{1}{2^9} )</td>
<td>0.0002170</td>
</tr>
<tr>
<td>( \frac{1}{11} \cdot \frac{1}{2^{11}} )</td>
<td>0.0000444</td>
</tr>
<tr>
<td>( \frac{1}{13} \cdot \frac{1}{2^{13}} )</td>
<td>0.0000094</td>
</tr>
</tbody>
</table>

\[ \text{Sum of positive terms} = 0.5493036 \]

### Negative terms

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} \cdot \frac{1}{2^2} )</td>
<td>0.125000</td>
</tr>
<tr>
<td>( \frac{1}{4} \cdot \frac{1}{2^4} )</td>
<td>0.0156250</td>
</tr>
<tr>
<td>( \frac{1}{6} \cdot \frac{1}{2^6} )</td>
<td>0.0026042</td>
</tr>
<tr>
<td>( \frac{1}{8} \cdot \frac{1}{2^8} )</td>
<td>0.0004883</td>
</tr>
<tr>
<td>( \frac{1}{10} \cdot \frac{1}{2^{10}} )</td>
<td>0.0000977</td>
</tr>
<tr>
<td>( \frac{1}{12} \cdot \frac{1}{2^{12}} )</td>
<td>0.0000203</td>
</tr>
<tr>
<td>( \frac{1}{14} \cdot \frac{1}{2^{14}} )</td>
<td>0.0000044</td>
</tr>
</tbody>
</table>

\[ \text{Sum of negative terms} = 0.1438399 \]

Subtracting the sum of the negative from the sum of the positive terms, we obtain

\[ \log \sqrt{\frac{3}{2}} = 0.4054637. \]

Denote the sum of the remaining terms of the series by \( R \). Then, by Algebra,

\[ R < \frac{1}{15} \cdot \frac{1}{2^{15}} \]

\[ < 0.000021. \]

The error caused by retaining no more decimal places in the computation is less than 0.000006. Hence the total error is less than 0.000027. Therefore the result is correct to five decimal places.

### 61

As remarked, the series (1) does not enable us to calculate directly the logarithms of numbers greater than 2, but it can be readily transformed into a series which gives us the logarithm of any positive number.

Replacing \( x \) by \(-x\) in (1), we obtain
\[ \log_e (1-x) = -x - \frac{x^3}{2} - \frac{x^5}{3} - \frac{x^7}{4} - \ldots \]

This series converges for \(-1 \leq x < 1\).

Subtracting this from (1), we obtain

\[ \log_e (1+x) - \log_e (1-x) = \log_e \left( \frac{1+x}{1-x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots \right), \tag{5} \]

which converges for \(-1 < x < 1\).

Putting \(y = \frac{1+x}{1-x}\), we see that \(y\) passes from 0 to \(\infty\) as \(x\) passes from \(-1\) to \(+1\); hence, if we make this substitution in (5), we get a series

\[ \log_e y = 2 \left[ \left( \frac{y-1}{y+1} \right) + \frac{1}{3} \left( \frac{y-1}{y+1} \right)^3 + \frac{1}{5} \left( \frac{y-1}{y+1} \right)^5 + \ldots \right], \tag{6} \]

which converges for all positive values of \(y\), and therefore enables us to compute the Naperian logarithm of any number.

From (5) we can get another series which is useful: put \(x = \frac{1}{2y+1}\); then, as \(\frac{1+x}{1-x} = \frac{y+1}{y}\), equation (5) gives us

\[ \log_e \left( \frac{y+1}{y} \right) = 2 \left( \frac{1}{2y+1} + \frac{1}{3} \cdot \frac{1}{(2y+1)^3} + \frac{1}{5} \cdot \frac{1}{(2y+1)^5} + \ldots \right), \]

which converges for all positive values of \(y\). Hence,

\[ \log_e (y+1) = \log_e y + 2 \left( \frac{1}{2y+1} + \frac{1}{3} \cdot \frac{1}{(2y+1)^3} + \frac{1}{5} \cdot \frac{1}{(2y+1)^5} + \ldots \right). \tag{7} \]

This series gives us \(\log_e (y+1)\), when \(\log_e y\) is known. It converges more rapidly than (6), when \(y\) is greater than 2, and hence should be used under these circumstances.

**62.** To construct a table we need to compute directly only the logarithms of prime numbers, since the others can be obtained by the relation

\[ \log xy = \log x + \log y. \]
Thus, to obtain the logarithms of the integers up to 10, we need to compute by series only the logarithms of the numbers 2, 3, 5, and 7.

(For \(4=2^2\), \(6=2 \cdot 3\), \(8=2^3\), \(9=3^2\), \(10=2 \cdot 5\), and \(\log 1=0\).)

In this case we are computing the logarithms of successive integers, and should therefore use (7).

**63. Example.**—Compute the Naperian logarithms of 2, 3, 4, and 5.

\[
\log_e 2 = 2 \left( \frac{1}{3} + \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^7} + \frac{1}{9} \cdot \frac{1}{3^9} + \ldots \right). \\
\frac{1}{3} = .333333 \\
\frac{1}{3} \cdot \frac{1}{3^3} = .0123457 \\
\frac{1}{5} \cdot \frac{1}{3^7} = .0008230 \\
\frac{1}{7} \cdot \frac{1}{3^9} = .0000653 \\
\frac{1}{9} \cdot \frac{1}{3^{3^3}} = .0000056 \\
\text{Denote the sum of the remaining terms of this series by } R. \\
\text{Then, by Algebra, } \\
R < \frac{1}{11} \cdot \frac{1}{3^n} \cdot \frac{1}{1-\frac{1}{3^2}} \\
or \ R < .000000573. \\
\text{The error caused by not retaining more places of decimals in the preceding column is less than } .0000005. \\
\text{Hence, the total error is less than } .00000165.
\]

**Remark.**—We should get the same series if we were to use (6).

\[
\log_e 3 = \log_e 2 + 2 \left( \frac{1}{5} + \frac{1}{5^3} + \frac{1}{5} \cdot \frac{1}{5^7} + \frac{1}{5} \cdot \frac{1}{5^{10}} + \ldots \right) \\
\frac{1}{5} = .2000000 \\
\frac{1}{5} \cdot \frac{1}{5^3} = .0026667 \\
\frac{1}{5} \cdot \frac{1}{5^7} = .0000640 \\
\frac{1}{7} \cdot \frac{1}{5^{10}} = .0000018 \\
\text{Denote the sum of the remaining terms of this series by } R. \\
\text{Then, by Algebra, } \\
R < \frac{1}{9} \cdot \frac{1}{5^9} \cdot \frac{1}{1-\frac{1}{5^2}} \\
or \ R < .0000006. \\
\text{Noting the errors in the preceding column and in } \log_e 2, \text{ we see that the total error is less than } .00000217.
\]
Remark.—If we were to use (6) to compute \( \log_e 3 \), we should have
\[
\log_e 3 = 2 \left[ \frac{1}{2} + \frac{1}{3} \left( \frac{1}{2} \right)^3 + \frac{1}{5} \left( \frac{1}{2} \right)^5 + \frac{1}{7} \left( \frac{1}{2} \right)^7 + \cdots \right].
\]

This series converges much more slowly than the above, since its terms are multiples of powers of \( \frac{1}{3} \), while the terms of the above are the same multiples of powers of \( \frac{1}{5} \). Thus, we should be obliged to use eight instead of four terms to have the result correct to five places.

\[
\log_e 4 = 2 \log_e 2 = 1.3862916.
\]

\[
\log_e 5 = \log_e 4 + 2\left( \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{9^3} + \frac{1}{5} \cdot \frac{1}{9^5} + \cdots \right),
\]
or \( \log_e 5 = 1.60944 \).

64. Proceeding in like manner, we may calculate any number of logarithms.

The following table gives the Naperian logarithms of the first ten integers:

\[
\begin{array}{c|c}
\log_e 1 & 0.00000 \\
\log_e 2 & 0.69315 \\
\log_e 3 & 1.09861 \\
\log_e 4 & 1.38629 \\
\log_e 5 & 1.60944 \\
\log_e 6 & 1.79176 \\
\log_e 7 & 1.94591 \\
\log_e 8 & 2.07944 \\
\log_e 9 & 2.19722 \\
\log_e 10 & 2.30259
\end{array}
\]

The common logarithm of any number may be found by multiplying its Naperian logarithm by \( M_{10} = 0.43429448 \).

Thus \( \log_{10} 5 = \log_e 5 \times 0.43429448 = 0.69897 \).

65. Remark.—If a table of logarithms were to be computed, the theory of interpolation and other special devices would be employed.

**COMPUTATION OF TRIGONOMETRIC FUNCTIONS**

66. Since \( \tan x = \frac{\sin x}{\cos x} \), \( \cot x = \frac{\cos x}{\sin x} \), etc., the computation of all the trigonometric functions depends upon that of the sine and cosine; thus the developments (2) and (3) suffice for all the trigonometric functions. Further, since the
sine or cosine of any angle is a sine or cosine of an angle
\( \leq \frac{\pi}{4} \), it is never necessary to take \( x \) greater than \( \frac{\pi}{4} \) in the
series (2) and (3).

Since \( \frac{\pi}{4} = 0.785398 \ldots < \frac{8}{10} \), these series converge rapidly; in fact,
\( \frac{1}{9!} = .000003 \) does not affect the fifth decimal place, and \( \frac{1}{11!} \) the
seventh.

**67. Remark.**—In the systematic computation of tables we should
not calculate the functions of each angle from the series independently. We should rather make use of the formulas (25) and (27) of § 38, thus obtaining

\[
\sin nx = 2 \cos x \sin (n - 1)x - \sin (n - 2)x,
\]
\[
\cos nx = 2 \cos x \cos (n - 1)x - \cos (n - 2)x.
\]

If our tables are to be at intervals of 1', we should calculate the
sine and cosine of 1' by the series. The above expressions then enable us to find successively the sine and cosine of 2', 3', 4', etc., till we have the sine and cosine of all angles up to 30° at intervals of 1'.

To obtain the sine and cosine of angles from 30° to 45° we should make use of these results by means of the formulas

\[
\sin (30° + y) = \cos y - \sin (30° - y),
\]
\[
\cos (30° + y) = \cos (30° - y) - \sin y.
\]

**68. To employ series (2) and (3) in computing the sine and cosine** we must first convert the angle into circular measure.

To do this we recall that

\[
1° = .017453293, \quad 1' = .0002908882, \quad 1'' = .000004848137.
\]

**Example.**—To compute the sine and cosine of 12° 15' 39''.

\[
12° = .209439516 \\
15' = .004363323 \\
39'' = .000189076 \\
12° 15' 39'' = .213991915 \text{ in circular measure.}
\]
\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
\]
\[x = .2139919\]
\[\frac{x^3}{3!} = .0000037\]
\[\frac{x^5}{5!} = .2139956\]
\[\text{subtract } \frac{x^3}{3!} = .0016332\]
\[\sin x = .2123624\]
Correct to five decimal places.

\[
\cos x = 1 - \frac{x}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}
\]
\[1 = 1.0000000\]
\[\frac{x^4}{4!} = .00000874\]
\[\text{subtract } \frac{x^4}{4!} = .0228963\]
\[\cos x = .9771911\]
Correct to five decimal places.

**DE MOIVRE’S THEOREM**

**69.** In Algebra we learn that the complex number
\[a = a + \beta \sqrt{-1} = a + \beta i\]  \hspace{1cm} (8)
may be represented graphically thus:

![Diagram](image)

Take two lines, \(OX\) and \(OY\), at right angles to each other. To the number \(a\) will correspond the point \(A\), whose distances from the two lines of reference are \(\beta\) and \(a\) respectively.

This geometrical representation shows at once that we can also write \(a\) in the form
\[a = r (\cos \theta + i \sin \theta).\]  \hspace{1cm} (9)

**70.** From Algebra we recall the definition of the sum of the complex numbers \(a = a + \beta i\) and \(b = \gamma + i \delta\); namely
\[a + b = a + \gamma + i(\beta + \delta).
\]
Subtraction is defined as the inverse of addition, so that
\[a - b = a - \gamma + i(\beta - \delta).\]
DE MOIVRE'S THEOREM

Multiplication is most conveniently defined when \( a \) and \( b \) are written in form (9). If
\[
a = r (\cos \varphi + i \sin \varphi) \quad \text{and} \quad b = s (\cos \phi + i \sin \phi),
\]
their product is defined by the equation
\[
ab = rs [\cos (\varphi + \phi) + i \sin (\varphi + \phi)]. \quad (10)
\]
Division is defined as the inverse of multiplication, so that
\[
\frac{a}{b} = \frac{r}{s} [\cos (\varphi - \phi) + i \sin (\varphi - \phi)].
\]
Finally, we recall that in an equation between complex numbers,
\[
a + i \beta = \gamma + i \delta,
\]
we have
\[
a = \gamma, \quad \beta = \delta. \quad (11)
\]

7.1. Consider the different powers of the complex number
\[
x = \cos \varphi + i \sin \varphi.
\]
By (10) we have
\[
x^2 = (\cos \varphi + i \sin \varphi) (\cos \varphi + i \sin \varphi),
\]
\[
= \cos 2\varphi + i \sin 2\varphi.
\]
\[
x^3 = x^2 \cdot x = (\cos 2\varphi + i \sin 2\varphi) (\cos \varphi + i \sin \varphi),
\]
\[
= \cos 3\varphi + i \sin 3\varphi.
\]
And, in general, for any integer \( n \),
\[
x^n = (\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi.
\]
From this equation we have De Moivre's Theorem, which is expressed by the formula
\[
(\cos \varphi + i \sin \varphi)^n = (\cos n\varphi + i \sin n\varphi). \quad (12)
\]

7.2. An interesting application of De Moivre's Theorem is the expansion of \( \sin nx \) and \( \cos nx \) in terms of \( \sin x \) and \( \cos x \). Expanding the left-hand side of (12) by the binomial theorem, and substituting \( x \) for \( \varphi \), we have
\[
\cos nx + i \sin nx = \cos^n x + n \cos^{n-1} x (i \sin x) + \frac{n(n-1)}{2!} \cos^{n-2} x
\]
\[
(i \sin x)^2 + \frac{n \cdot (n-1)(n-2)}{3!} \cos^{n-3} x (i \sin x)^3 + \ldots
\]
or
\[
\cos nx + i \sin nx = \left( \cos^n x - \frac{n(n-1)}{2!} \cos^{n-2} x \sin^2 x + \ldots \right)
+ i \left[ n \cos^{n-1} x \sin x - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} x \sin^3 x + \ldots \right].
\]

Equating real and imaginary parts, as in (11), we have
\[
\cos nx = \cos^n x - \frac{n(n-1)}{2!} \cos^{n-2} x \sin^2 x + \ldots \tag{13}
\]
\[
\sin nx = n \cos^{n-1} x \sin x - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} x \sin^3 x + \ldots \tag{14}
\]

Example. \( n = 5 \).
\[
\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x.
\]
\[
\sin 5x = 5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x.
\]

### THE ROOTS OF UNITY

### 73. We find another application of De Moivre's Theorem in obtaining the roots of unity. The \( n \)th roots of unity are by definition the roots of the equation
\[
x^n = 1.
\]

Every equation has \( n \) roots and no more; hence, if we can find \( n \) distinct numbers which satisfy this equation we shall have all the \( n \)th roots of unity.

Consider the \( n \) numbers
\[
x_r = \cos \frac{2\pi r}{n} + i \sin \frac{2\pi r}{n},
\]
\( r = 0, 1, 2, \ldots n-1. \)

Geometrically these numbers are represented by the \( n \) vertices of a regular polygon. They are, therefore, all different. We shall see now that they are precisely the \( n \)th roots of unity.

In fact, we have by (12),
\[
x^n_r = \left( \cos \frac{2\pi r}{n} + i \sin \frac{2\pi r}{n} \right)^n,
\]
\[ x_{r} = \cos \left( n \cdot \frac{2\pi r}{n} \right) + i \sin \left( n \cdot \frac{2\pi r}{n} \right), \]
\[ = \cos 2\pi r + i \sin 2\pi r, \]
\[ = 1 + i \cdot 0 = 1. \]

Therefore \( x_{r} \) is one of the roots of unity.

Thus the cube roots of unity are represented by the points \( A, P, \) and \( Q \) of the following figure. In the figure \( OA = 1, \) angle \( AOP = \frac{2\pi}{3} = 120^\circ, \) angle \( AOQ = \frac{4\pi}{3} = 240^\circ; \) that is, the circumference is divided into three equal parts by the points \( A, P, \) and \( Q. \) Then \( OD = \frac{1}{2}, \) and \( DP = DQ = \frac{1}{2} \sqrt{3}. \) Hence we see from the method of representing a complex number given above that \( A \) represents \( +1, \) \( P \) represents \(-\frac{1}{2} + i \frac{\sqrt{3}}{2}, \) \( Q \) represents \(-\frac{1}{2} - i \frac{\sqrt{3}}{2}. \)

![Diagram](image)

**EXERCISES**

74. (1.) Express \( \sin 4x \) and \( \cos 4x \) in terms of \( \sin x \) and \( \cos x. \)
(2.) Express \( \sin 6x \) and \( \cos 6x \) in terms of \( \sin x \) and \( \cos x. \)
(3.) Find the six 6th roots of unity.
(4.) Find the five 5th roots of unity.

**THE HYPERBOLIC FUNCTIONS**

75. The hyperbolic functions are defined by the equations

\[ \sinh x = \frac{e^x - e^{-x}}{2}, \quad (15) \]
\[ \cosh x = \frac{e^x + e^{-x}}{2}, \quad (16) \]

in which \( \sinh x \) and \( \cosh x \) denote the hyperbolic sine and
hyperbolic cosine of \( x \) respectively. These functions are called the hyperbolic sine and cosine on account of their relation to the hyperbola analogous to the relation of the sine and cosine to the circle. A natural and convenient way to arrive at the hyperbolic functions and to study their properties is by using complex numbers in the following manner. The series (2), (3), and (4) give the value of \( \sin x \), \( \cos x \), and \( e^x \) for every real value of \( x \). These series also serve to define \( \sin x \), \( \cos x \), and \( e^x \) for complex values of \( x \). In the more advanced parts of Algebra it is shown that the following fundamental formulas which we have proved only for a real variable,

\[
\begin{align*}
\sin (x + y) &= \sin x \cos y + \cos x \sin y, \\
\cos (x + y) &= \cos x \cos y - \sin x \sin y, \\
e^{x+y} &= e^x e^y,
\end{align*}
\]

hold unchanged when the variable is complex.

This fact enables us to calculate with ease \( \sin x \), \( \cos x \), and \( e^x \) for any complex value of the variable.

In so doing we are led directly to the hyperbolic functions. At the same time a relation between the trigonometric and hyperbolic functions is established by means of which the formulas of Chapter III. can be converted into corresponding formulas for the hyperbolic functions.

Taking \( x \) and \( y \) real and replacing \( y \) in (17), (18), and (19) by \( iy \), we get

\[
\begin{align*}
\sin (x + iy) &= \sin x \cos iy + \cos x \sin iy, \\
\cos (x + iy) &= \cos x \cos iy - \sin x \sin iy, \\
e^{x+iy} &= e^x e^{iy}.
\end{align*}
\]

Thus the calculation of these functions when the variable is complex is made to depend upon the case where the variable is a pure imaginary.
HYPERBOLIC FUNCTIONS

If we replace $x$ by $ix$ in series (4) we obtain

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).$$

A comparison with series (2) and (3) shows that these two series are $\cos x$ and $\sin x$ respectively; hence the important formula due to Euler—

$$e^{ix} = \cos x + i \sin x. \quad (20)$$

This enables us to calculate $e^{ix}$ from $\sin x$ and $\cos x$ when $ix$ is a pure imaginary; that is, when $x$ is real.

To find $\sin ix$ and $\cos ix$ replace $x$ in (20) by $ix$; we obtain

$$e^{-x} = \cos ix + i \sin ix. \quad (21)$$

Again replacing $x$ by $-ix$ in (20), we obtain

$$e^x = \cos ix - i \sin ix. \quad (22)$$

The sum and difference of (21) and (22) give

$$\cos ix = \frac{e^x + e^{-x}}{2} = \cosh x, \quad (23)$$

$$\sin ix = \frac{i(e^x - e^{-x})}{2} = i \sinh x. \quad (24)$$

If we compute the value of $e^x$ by the aid of series (4) for a succession of values of $x$, we find that $\sinh x$ and $\cosh x$ are represented by the curves on page 76.

The system of formulas belonging to the hyperbolic functions is obtained from those of the trigonometric functions by using (23) and (24). This shows that for every formula in analytic trigonometry there exists a corresponding formula in hyperbolic trigonometry which we get by this sub-
stitution. In the examples which follow, this method is used to obtain important formulas in hyperbolic trigonometry.

Replacing \(x\) by \(-ix\) in (23) and (24), we get

\[
\begin{align*}
\cos x &= \frac{e^{ix} + e^{-ix}}{2}, \\
\sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{align*}
\]

which are formulas frequently used.

**Example.** — \(\sinh (x + y) = -i \sin i(x + y),\)

\[
= -i [\sin ix \cos iy + \cos ix \sin iy],
\]

\[
= -i [i \sinh x \cosh y + i \cosh x \sinh y],
\]

\[
= \sinh x \cosh y + \cosh x \sinh y.
\]

**Example.** — \(\sinh x + \sinh y = -i (\sin ix + \sin iy),\)

\[
= -i 2 \sin \frac{1}{2} i(x + y) \cos \frac{1}{2} i(x - y),
\]

\[
= 2 \sinh \frac{1}{2} (x + y) \cosh \frac{1}{2} (x - y).
\]
EXERCISES

76. (1.) Prove \( \sinh 0 = 0, \ \cosh 0 = 1 \).

(2.) Prove \( \sinh \frac{1}{2} \pi i = i, \ \cosh \frac{1}{2} \pi i = 0 \).

(3.) Prove \( \sinh \pi i = 0, \ \cosh \pi i = -1 \).

Prove that

(4.) \( \sin (-ix) = -\sin ix \).

(5.) \( \cos (-ix) = \cos ix \).

(6.) \( \sinh (-x) = -\sinh x \).

(7.) \( \cosh (-x) = \cosh x \).

Remark.—The hyperbolic tangent, cotangent, secant, and cosecant are defined by

\[
\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x},
\]

\[
\text{sech} \ x = \frac{1}{\cosh x}, \quad \text{csch} \ x = \frac{1}{\sinh x}.
\]

Prove that

(8.) \( \tan (ix) = i \tanh x \).

(9.) \( \coth (-x) = -\coth x \).

(10.) \( \text{sech} (-x) = \text{sech} x \).

(11.) \( \cosh^2 x - \sinh^2 x = 1 \).

(12.) \( \text{sech}^2 x + \tanh^2 x = 1 \).

(13.) \( \coth^2 x - \text{csch}^2 x = 1 \).

(14.) \( \sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y \).

(15.) \( \cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y \).

(16.) \( \cosh \frac{1}{2} x = \sqrt{\frac{1 + \cosh x}{2}} \).

(17.) \( \sinh u - \sinh v = 2 \cosh \frac{1}{2} (u + v) \sinh \frac{1}{2} (u - v) \).

(18.) \( \cosh u + \cosh v = 2 \cosh \frac{1}{2} (u + v) \cosh \frac{1}{2} (u - v) \).

(19.) \( \cosh u - \cosh v = 2 \sinh \frac{1}{2} (u + v) \sinh \frac{1}{2} (u - v) \).
CHAPTER VII

MISCELLANEOUS EXERCISES

RELATION OF FUNCTIONS

77. Prove the following:

1. \( \cos x = \sin x \cot x \).

2. \( \csc x \tan x = \sec x \).

3. \( (\tan x + \cot x) \sin x \cos x = 1 \).

4. \( (\sec y - \tan y)(\sec y + \tan y) = 1 \).

5. \( (\csc z - \cot z)(\csc z + \cot z) = 1 \).

6. \( \cos^2 y + (\tan y - \cot y) \sin y \cos y = \sin^2 y \).

7. \( \cos^4 x - \sin^4 x + 1 = 2 \cos^2 x \).

8. \( (\sin y - \cos y)^2 = 1 - 2 \sin y \cos y \).

9. \( \sin^2 x + \cos^2 x = (\sin x + \cos x)(1 - \sin x \cos x) \).

10. \( \frac{\cot x + \tan y}{\tan x + \cot y} = \cot x \tan y \).

11. \( \cos^2 y - \sin^2 y = 2 \cos^2 y - 1 \).

12. \( 1 - \tan^4 x = 2 \sec^2 x - \sec^4 x \).

13. \( \frac{\cos x}{\sin x \cot^2 x} = \tan x \).

14. \( \sec^2 y \csc^2 y = \tan^2 y + \cot^2 y + 2 \).

15. \( \cot y - \csc y \sec y (1 - 2 \sin^2 y) = \tan y \).

16. \( \left( \frac{1}{\sin z} - \cot z \right)^2 = \frac{1 - \cos z}{1 + \cos z} \).

17. \( \frac{\sec y}{1 + \cos y} = \frac{\tan y - \sin y}{\sin^3 y} \).

18. \( 1 + \frac{2 \sin x}{\sec x} = (\sin x + \cos x)^2 \).

19. \( \frac{1}{\sec^3 x} - \sin^3 x = (\cos x - \sin x)(1 + \sin x \cos x) \).

20. \( (\sin x \cos y + \cos x \sin y)^2 + (\cos x \cos y - \sin x \sin y)^2 = 1 \).
MISCELLANEOUS EXERCISES

(21.) \((a \cos x - b \sin x)^2 + (a \sin x + b \cos x)^2 = a^2 + b^2\).

(22.) \(\frac{1}{(\cos^2 y - \sin^2 y)^2} = 1 + \frac{4 \tan^2 y}{(1 - \tan^2 y)^2}\).

Find an angle not greater than 90° which satisfies each of the following equations:

(23.) \(4 \cos x = 3 \sec x\).

(24.) \(\sin y = \csc y - \frac{3}{4}\).

(25.) \(\sqrt{2} \sin x - \tan x = 0\).

(26.) \(2 \cos x - \sqrt{3} \cot x = 0\).

(27.) \(\tan y + \cot y - 2 = 0\).

(28.) \(2 \sin^2 y - 2 = -\sqrt{2} \cos y\).

(29.) \(3 \tan^2 x - 1 = 4 \sin^2 x\).

(30.) \(\cos^2 x + 2 \sin^2 x - \frac{5}{2} \sin x = 0\).

(31.) \(\csc x = \frac{4}{3} \tan x\).

(32.) \(\sec x + \tan x = \pm \sqrt{3}\).

(33.) \(\tan x + 2 \sqrt{3} \cos x = 0\).

(34.) \(3 \sin x - 2 \cos^2 x = 0\).

Express the following in terms of the functions of angles less than 45°:

(35.) \(\sin 92°\).

(36.) \(\cos 127°\).

(37.) \(\tan 320°\).

(38.) \(\cot 350°\).

(39.) \(\sin 265°\).

(40.) \(\tan 171°\).

(41.) Given \(\sin x = \frac{4}{5}\) and \(x\) in quadrant II; find all the other functions of \(x\).

(42.) Given \(\cos x = -\frac{3}{5}\) and \(x\) in quadrant III; find all the other functions of \(x\).

(43.) Given \(\tan x = \frac{3}{4}\) and \(x\) in quadrant III; find all the other functions of \(x\).

(44.) Given \(\cot x = -\frac{3}{2}\) and \(x\) in quadrant IV; find all the other functions of \(x\).
In what quadrants must the angles lie which satisfy each of the following equations:

(45.) \( \sin x \cos x = -\frac{1}{2} \sqrt{3} \).
(46.) \( \sec x \tan x = 2 \sqrt{3} \).
(47.) \( \tan y + \sqrt{2} \cos y = 0 \).
(48.) \( \cos x \cot x = \frac{2}{3} \).

Find all the values of \( y \) less than 360° which will satisfy the following equations:

(49.) \( \tan y + 2 \sin y = 0 \).
(50.) \( (1 + \tan x)(1 - 2 \sin x) = 0 \).
(51.) \( \sin x \cos x (1 + 2 \cos x) = 0 \).

Prove the following:

(52.) \( \cos 780° = \frac{1}{2} \).
(53.) \( \sin 1485° = \frac{1}{2} \sqrt{2} \).
(54.) \( \cos 2550° = \frac{1}{2} \sqrt{3} \).
(55.) \( \sin (-3000°) = -\cos 30° \).
(56.) \( \cos 1300° = -\cos 40° \).

(57.) Find the value of \( a \sin 90° + b \tan 0° + a \cos 180° \).
(58.) Find the value of \( a \sin 30° + b \tan 45° + a \cos 60° + b \tan 135° \).
(59.) Find the value of \((a - b) \tan 225° + b \cos 180° - a \sin 270° \).
(60.) Find the value of \((a \sin 45° + b \cos 45°)(a \sin 135° + b \sin 225°) \).

RIGHT TRIANGLES

78. In the following problems the planes on which distances are measured are understood to be horizontal unless otherwise stated.

(1.) The angle of elevation of the top of the tower from a point 1121 ft. from its base is observed to be 15° 17'; find the height of the tower.

(2.) A tree, 77 ft. high, stands on the bank of a river; at a point on the other bank just opposite the tree the angle of elevation of the top of the tree is found to be 5° 17' 37". Find the breadth of the river.
(3.) What angle will a ladder 42 ft. long make with the ground if its foot is 25 ft. from the base of the building against which it is placed?

(4.) When the altitude of the sun is 33° 22', what is the height of a tree which casts a shadow 75 ft.?

(5.) Two towns are 3 miles apart. The angle of depression of one, from a balloon directly above the other, is observed to be 8° 15'. How high is the balloon?

(6.) From a point 197 ft. from the base of a tower the angle of elevation was found to be 46° 45' 54''; find the height of the tower.

(7.) A man 5 ft. 10 in. high stands at a distance of 4 ft. 7 in. from a lamp-post, and casts a shadow 18 ft. long; find the height of the lamp-post.

(8.) The shadow of a building 101.3 ft. high is found to be 131.5 ft. long; find the elevation of the sun at that time.

(9.) A rope 112 ft. long is attached to the top of a building and reaches the ground, making an angle of 77° 20' with the ground; find the height of the building.

(10.) A house is 130 ft. above the water, on the banks of a river; from a point just opposite on the other bank the angle of elevation of the house is 14° 30' 21''. Find the width of the river.

(11.) From the top of a headland, 1217.8 ft. above the level of the sea, the angle of depression of a dock was observed to be 10° 9' 13''; find the distance from the foot of the headland to the dock.

(12.) 1121.5 ft. from the base of a tower its angle of elevation is found to be 11° 3' 5''; find the height of the tower.

(13.) One bank of a river is 94.73 ft. vertically above the water, and subtends an angle of 10° 54' 13'' from a point directly opposite at the water's edge; find the width of the river.

(14.) The shadow of a vertical cliff 113 ft. high just reaches a boat on the sea 93 ft. from its base; find the altitude of the sun.

(15.) A rope, 38 ft. long, just reached the ground when fastened to the top of a tree 29 ft. high. What angle does it make with the ground?

(16.) A tree is broken by the wind. Its top strikes the ground 15 ft. from the foot of the tree, and makes an angle of 42° 28' with the ground. Find the height of the tree before it was broken.
(17.) The pole of a circular tent is 18 ft. high, and the ropes reaching from its top to stakes in the ground are 37 ft. long; find the distance from the foot of the pole to one of the stakes, and the angle between the ground and the ropes.

(18.) A ship is sailing southwest at the rate of 8 miles an hour. At what rate is it moving south?

(19.) A building is 121 ft. high. From a point directly across the street its angle of elevation is 65° 3'. Find the width of the street.

(20.) From the top of a building 52 ft. high the angle of elevation of another building 112 ft. high is 30° 12'. How far are the buildings apart?

(21.) A window in a house is 24 ft. from the ground. What is the inclination of a ladder placed 8 ft. from the side of the building and reaching the window?

(22.) Given that the sun’s distance from the earth is 92,000,000 miles, and its apparent semidiameter is 16' 2"; find its diameter.

(23.) Given that the radius of the earth is 3963 miles, and that it subtends an angle of 57' 2" at the moon; find the distance of the moon from the earth.

(24.) Given that when the moon’s distance from the earth is 238885 miles, its apparent semidiameter is 15' 34"; find its diameter in miles.

(25.) Given that the radius of the earth is 3963 miles, and that it subtends an angle of 9" at the sun; find the distance of the sun from the earth.

(26.) A light-house is 57 ft. high; the angles of elevation of the top and bottom of it, as seen from a ship, are 5° 3' 20" and 4° 28' 8". Find the distance of its base above the sea-level.

(27.) At a certain point the angle of elevation of a tower was observed to be 53° 51' 16", and at a point 302 ft. farther away in the same straight line it was 9° 52' 10"; find the height of the tower.

(28.) A tree stands at a distance from a straight road and between two mile-stones. At one mile-stone the line to the tree is observed to make an angle of 25° 15' with the road, and at the other an angle of 45° 17'. Find the distance of the tree from the road.

(29.) From the top of a light-house, 225 ft. above the level of the sea, the angles of depression of two ships are 17° 21' 50" and 13° 50' 22",
and the line joining the ships passes directly beneath the light-house: find the distance between the two ships.

**ISOSCELES TRIANGLES AND REGULAR POLYGONS**

79. (1.) The area of a regular dodecagon is 37.52 ft.; find its apothem.

(2.) The perimeter of a regular polygon of 11 sides is 23.47 ft.; find the radius of the circumscribing circle.

(3.) A regular decagon is circumscribed about a circle whose radius is 3.147 ft.; find its perimeter.

(4.) The side of a regular decagon is 23.41 ft.; find the radius of the inscribed circle.

(5.) The perimeter of an equilateral triangle is 17.2 ft.; find the area of the inscribed circle.

(6.) The area of a regular octagon is 2478 sq. in.; find its perimeter.

(7.) The area of a regular pentagon is 32.57 sq. ft.; find the radius of the inscribed circle.

(8.) The angle between the legs of a pair of dividers is 43°, and the legs are 7 in. long; find the distance between the points.

(9.) A building is 37.54 ft. wide, and the slope of the roof is 43° 36'; find the length of the rafters.

(10.) The radius of a circle is 12732, and the length of a chord is 18321; find the angle the chord subtends at the centre.

(11.) If the radius of a circle is taken as unity, what is the length of a chord which subtends an angle of 77° 17' 40''?

(12.) What angle at the centre of a circle does a chord which is $\frac{1}{4}$ of the radius subtend?

(13.) What is the radius of a circle if a chord 11223 ft. subtends an angle of 50° 50' 52''?

(14.) Two light-houses at the mouth of a harbor are each 2 miles from the wharf. A person on the wharf finds the angle between the lines to the light-houses to be 17° 32'. Find the distance between the two light-houses.

(15.) The side of a regular pentagon is 2; find the radius of the inscribed circle.
(16.) The perimeter of a regular heptagon inscribed in a circle is 12; find the radius of the circle.

(17.) The radius of a circle inscribed in an octagon is 3; find the perimeter of the octagon.

(18.) A regular polygon of 9 sides is inscribed in a circle of unit radius; find the radius of the inscribed circle.

(19.) Find the perimeter of a regular decagon circumscribed about a unit circle.

(20.) Find the area of a regular hexagon circumscribed about a unit circle.

(21.) Find the perimeter of a polygon of 11 sides inscribed in a unit circle.

(22.) The perimeter of a dodecagon is 30; find its area.

(23.) The area of a regular polygon of 11 sides is 18; find its perimeter.

TRIGONOMETRIC IDENTITIES AND EQUATIONS

80. Prove the following:

1. \( \sin \frac{1}{2} y \pm \cos \frac{1}{2} y = \sqrt{1 \pm \sin y} \).

2. \( \frac{\cos x - \cos y}{\cos x + \cos y} = -\tan \frac{1}{2} (x + y) \tan \frac{1}{2} (x - y) \).

3. \( \frac{\sin 2x + \sin 4x}{\cos 2x + \cos 4x} = \tan 3x \).

4. \( \cos^2 y \tan^2 y + \sin^2 y \cot^2 y = 1 \).

5. \( \frac{\cos (x + y + z)}{\sin x \sin y \sin z} = \cot x \cot y \cot z - \cot x - \cot y - \cot z \).

6. \( \cos^2 (x - y) - \sin^2 (x + y) = \cos 2x \cos 2y \).

7. \( \frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{1}{2} (x - y) \).

8. \( \frac{\cos x - \sec x}{\sec x} = 4 \cos^2 \frac{1}{2} x (\cos^2 \frac{1}{2} x - 1) \).

9. \( \cot x = \frac{\sin 2x}{1 - \cos 2x} \).

10. \( \tan^2 y = \frac{1 - \cos 2y}{1 + \cos 2y} \).

11. \( \cot x - \tan x = 2 \cot 2x \).
(12.) \[ \tan \frac{1}{2} x + 2 \sin^2 \frac{1}{2} x \cot x = \sin x. \]

(13.) \[ \tan x \pm \tan y \over \cot x \pm \cot y = \pm \sin x \sec x \tan y. \]

(14.) \[ \sin x - 2 \sin^2 x = \sin x \cos 2x. \]

(15.) \[ 4 \sin y \sin (60^\circ - y) \sin (60^\circ + y) = \sin 3y. \]

(16.) \[ \frac{\sin y (1 - \tan^2 y)}{\sec^2 y} \left( {\sec y - \sin y \over \cos y \sin y} + {\sec y + \sin y \over \cos y \sin y} \right) = \sin 2y. \]

(17.) \[ 1 + \tan y \tan \frac{1}{2} y = \sec y. \]

(18.) \[ \sin 4x = 4 \sin x \cos^2 x - 4 \cos x \sin^3 x. \]

(19.) \[ \sec 2x + \tan 2x + 1 = \frac{2}{1 - \tan x}. \]

(20.) \[ \tan 50^\circ + \cot 50^\circ = 2 \sec 10^\circ. \]

(21.) \[ \cos (x + 45^\circ) + \sin (x - 45^\circ) = 0. \]

(22.) \[ {\tan x \over 1 - \cot 2x \tan x} = \sin 2x. \]

(23.) \[ (1 - \tan^2 x) \sin x \cos x = \cos 2x \sqrt{1 - \cos 2x \over 1 + \cos 2x}. \]

(24.) \[ \cos y + \sin y \over \cos y - \sin y = \tan 2y + \sec 2y. \]

(25.) \[ \sin (x + y) \cos x \cos (x + y) \sin x = \sin y. \]

(26.) \[ \cos (x - y) \sin y + \sin (x - y) \cos y = \sin x. \]

(27.) \[ {\sin (x - y) \over \cos x \cos y} + {\sin (y - z) \over \cos y \cos z} + {\sin (z - x) \over \cos z \cos x} = 0. \]

(28.) \[ {\sin x + \sin 2x \over \cos x - \cos 2x} = \cot \frac{1}{2} x. \]

(29.) \[ 2 \sin^2 x \sin^2 y + 2 \cos^2 x \cos^2 y = 1 + \cos 2x \cos 2y. \]

(30.) \[ \sin 60^\circ + \sin 30^\circ = 2 \sin 45^\circ \cos 15^\circ. \]

(31.) \[ {\tan (x - y) + \tan y \over 1 - \tan (x - y) \tan y} = \tan x. \]

(32.) \[ 2 \over \sin y \tan \frac{1}{2} y = 1 + \cot^2 \frac{1}{2} y. \]

(33.) \[ \sin 4x + \sin 2x = 2 \sin 3x \cos x. \]

(34.) \[ \sin x + \sin y \over \cos x - \cos y = \cos x + \cos y \over \sin y - \sin x. \]

(35.) \[ \sin 75^\circ = \sqrt{3 + 1 \over 2 \sqrt{2}}. \]

(36.) \[ 2 \tan 2y = \tan (45^\circ + y) - \tan (45^\circ - y). \]
(37.) \[ \frac{\tan 2x + \tan x}{\tan 2x - \tan x} = \frac{\sin 3x}{\sin x}. \]

(38.) \[ \tan 3y = \frac{3 \tan y - \tan^3 y}{1 - 3 \tan^2 y}. \]

(39.) \[ \sin 60^\circ + \sin 20^\circ = 2 \sin 40^\circ \cos 20^\circ. \]

(40.) \[ \sin 40^\circ - \sin 10^\circ = 2 \cos 25^\circ \sin 15^\circ. \]

(41.) \[ \cos 2x - \cos 4x = 2 \sin 3x \sin x. \]

(42.) \[ \tan 15^\circ = 2 - \sqrt{3}. \]

(43.) \[ (\sqrt{1 + \sin x} - \sqrt{1 - \sin x})^2 = 4 \sin^2 \frac{x}{2}. \]

(44.) \[ (\sqrt{1 + \sin x} + \sqrt{1 - \sin x})^2 = 4 \cos^2 \frac{x}{2}. \]

(45.) \[ \frac{\sin(2x + y)}{\sin x} - 2 \cos(x + y) = \frac{\sin y}{\sin x}. \]

(46.) \[ \frac{\sin 4x}{\sin 2x} = 2 \cos 2x. \]

(47.) \[ \sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0. \]

(48.) \[ \cos \frac{\pi}{3} - \cos \frac{\pi}{2} = 2 \sin \frac{5\pi}{12} \sin \frac{\pi}{12}. \]

(49.) \[ \frac{1 - \tan^2(45^\circ - x)}{1 + \tan^2(45^\circ - x)} = \sin 2x. \]

(50.) \[ \frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \sqrt{\frac{1}{3}}. \]

(51.) \[ \tan^3 \frac{x}{2}(1 + \cot^2 \frac{x}{2})^2 = \frac{8}{\sin^2 x}. \]

(52.) \[ \tan 75^\circ = 2 + \sqrt{3}. \]

(53.) \[ \sin 3x + \sin 5x = 2 \sin 4x \cos x. \]

(54.) \[ \cos 5x + \cos 9x = 2 \cos 7x \cos 2x. \]

(55.) \[ \sin 15^\circ = \frac{\sqrt{3} - 1}{2 \sqrt{2}}. \]

(56.) \[ \frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x. \]

(57.) \[ \sin 5y = 5 \sin y - 20 \sin^3 y + 16 \sin^5 y. \]

(58.) \[ \cos 5y = 5 \cos y - 20 \cos^3 y + 16 \cos^5 y. \]

(59.) \[ \sin 4x = \frac{4 \tan x (1 - \tan^2 x)}{(1 + \tan^2 x)^2}. \]

(60.) \[ \cos(45^\circ + x) + \cos(45^\circ - x) = \sqrt{2} \cos x. \]

(61.) \[ \cos 3x + \cos 5x + \cos 7x + \cos 15x = 4 \cos 4x \cos 5x \cos 6x. \]
(62.) \(\sin^2 \frac{1}{2} x (\cot \frac{1}{2} x - 1)^2 = 1 - \sin x.\)

(63.) \(\frac{3\sin x - \sin 3x}{\cos 3x + 3 \cos x} = \tan^3 x.\)

(64.) \(\sin x (1 + \tan x) + \cos x (1 + \cot x) = \csc x + \sec x.\)

(65.) \(\frac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = \frac{2 + \sin 2x}{2}.\)

(66.) \(\cos y + \cos (120 - y) + \cos (120 + y) = 0.\)

(67.) \(\frac{\sin 3x}{\sin x} = 2 \cos 2x + 1.\)

(68.) \(\frac{(\cos y - \cos 3y)(\sin 8y + \sin 2y)}{(\sin 5y - \sin y)(\cos 4y - \cos 6y)} = 1.\)

(69.) \(\frac{\sin^2 x}{1 + \cos x} = \frac{1 - \cos x}{1 + \cos x}.\)

(70.) \(\frac{\sin 3x - \cos 3x}{\sin x \cos x} = 2.\)

(71.) \(\frac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot \frac{1}{2} x.\)

(72.) \(\frac{\sin (4x - 2y) + \sin (4y - 2x)}{\cos (4x - 2y) + \cos (4y - 2x)} = \tan (x + y).\)

(73.) \(\frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x.\)

If \(A, B,\) and \(C\) are the angles of a triangle, prove the following:

(74.) \(\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.\)

(75.) \(\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.\)

(76.) \(\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.\)

(77.) \(\tan A + \tan B + \tan C = \tan A \tan B \tan C.\)

Solve the following equations for values of \(x\) less than 360°.

(78.) \(\cos 2x + \cos x = -1.\)

(79.) \(\sin x + \sin 7x = \sin 4x.\)

(80.) \(\cos x - \sin 2x - \cos 3x = 0.\)

(81.) \(\cos x - \sin 3x - \cos 2x = 0.\)

(82.) \(\sin 4x - 2 \sin 2x = 0.\)

(83.) \(\sin 2x - \cos 2x - \sin x + \cos x = 0.\)

(84.) \(\sin (60° - x) - \sin (60° + x) = -\frac{1}{2} \sqrt{3}.\)

(85.) \(\sin (30° + x) - \cos (60° + x) = -\frac{1}{2} \sqrt{3}.\)
(86.) \(\csc x = 1 + \cot x\).

(87.) \(\cos 2x = \cos^2 x\).

(88.) \(2 \sin y = \sin 2y\).

(89.) \(\sin 3y + \sin 2y + \sin y = 0\).

(90.) \(\sin^2 x + 5 \cos^2 x = 3\).

(91.) \(\tan (45° - x) + \cot (45° - x) = 4\).

**OBLIQUE TRIANGLES**

81. (1.) It is required to find the distance between two points, \(A\) and \(B\), on opposite sides of a river. A line, \(AC\), and the angles \(BAC\) and \(ACB\) are measured and found to be 2483 ft., 61° 25', and 52° 17' respectively.

(2.) A straight road leads from a town \(A\) to a town \(B\), 12 miles distant; another road, making an angle of 77° with the first, goes from \(A\) to a town \(C\), 7 miles distant. How far are the towns \(B\) and \(C\) apart?

(3.) In order to determine the distance of a fort, \(A\), from a battery, \(B\), a line, \(BC\), one-half mile long, is measured, and the angles \(ABC\) and \(ACB\) are observed to be 75° 18' and 78° 21' respectively. Find the distance \(AB\).

(4.) Two houses, \(A\) and \(B\), are 1728 ft. apart. Find the distance of a third house, \(C\), from \(A\) if \(BAC = 47° 51'\) and \(ABC = 57° 23'\).

(5.) In order to determine the distance of a bluff, \(A\), from a house, \(B\), in a plane, a line, \(BC\), was measured and found to be 1281 yards, also the angles \(ABC\) and \(BCA\) 65° 31' and 70° 2' respectively. Find the distance \(AB\).

(6.) Two towns, 3 miles apart, are on opposite sides of a balloon. The angles of elevation of the balloon are found to be 13° 19' and 20° 3'. Find the distance of the balloon from the nearer town.

(7.) It is required to find the distance between two posts, \(A\) and \(B\), which are separated by a swamp. A point \(C\) is 1272.5 ft. from \(A\), and 2012.4 ft. from \(B\). The angle \(ACB\) is 41° 9' 11''.

(8.) Two stakes, \(A\) and \(B\), are on opposite sides of a stream; a third point, \(C\), is so situated that the distances \(AC\) and \(BC\) can be found, and are 431.27 yards and 601.72 yards respectively. The angle \(ACB\) is 39° 53' 15''. Find the distance between the stakes \(A\) and \(B\).
(9.) Two light-houses, \( A \) and \( B \), are 11 miles apart. A ship, \( C \), is observed from them to make the angles \( BAC = 31^\circ 13' 31'' \) and \( ABC = 21^\circ 46' 8'' \). Find the distance of the ship from \( A \).

(10.) Two islands, \( A \) and \( B \), are 6103 ft. apart. Find the distance from \( A \) to a ship, \( C \), if the angle \( ABC \) is \( 37^\circ 25' \) and \( BAC \) is \( 40^\circ 32' \).

(11.) In ascending a cliff towards a light-house at its summit, the light-house subtends at one point an angle of \( 21^\circ 22' \). At a point 55 ft. farther up it subtends an angle of \( 40^\circ 27' \). If the light-house is 58 ft. high, how far is this last point from its foot?

(12.) The distances of two islands from a buoy are 3 and 4 miles respectively. The islands are 2 miles apart. Find the angle subtended by the islands at the buoy.

(13.) The sides of a triangle are 151.45, 191.32, and 250.91. Find the length of the perpendicular from the largest angle upon the opposite side.

(14.) A tree stands on a hill, and the angle between the slope of the hill and the tree is \( 110^\circ 23' \). At a point 85.6 ft. down the hill the tree subtends an angle of \( 22^\circ 22' \). Find the height of the tree.

(15.) A light-house 54 ft. high is built upon a rock. From the top of the light-house the angle of depression of a boat is \( 19^\circ 10' \), and from its base the angle of depression of the boat is \( 12^\circ 22' \). Find the height of the rock on which the light-house stands.

(16.) Three towns, \( A \), \( B \), and \( C \), are connected by straight roads. \( AB = 4 \) miles, \( BC = 5 \) miles, and \( AC = 7 \) miles. Find the angle made by the roads \( AB \) and \( BC \).

(17.) Two buoys, \( A \) and \( B \), are one-half mile apart. Find the distance from \( A \) to a point \( C \) on the shore if the angles \( ABC \) and \( BAC \) are \( 77^\circ 7' \) and \( 67^\circ 17' \) respectively.

(18.) The top of a tower is 175 ft. above the level of a bay. From its top the angles of depression of the shores of the bay in a certain direction are \( 57^\circ 16' \) and \( 15^\circ 2' \). Find the distance across the bay.

(19.) The lengths of two sides of a triangle are \( \sqrt{2} \) and \( \sqrt{3} \). The angle between them is \( 45^\circ \). Find the remaining side.

(20.) The sides of a parallelogram are 172.43 and 101.31, and the angle included by them is \( 61^\circ 16' \). Find the two diagonals.

(21.) A tree 41 ft. high stands at the top of a hill which slopes
$10^\circ 12'$ to the horizontal. At a certain point down the hill the tree subtends an angle of $28^\circ 29'$. Find the distance from this point to the foot of the tree.

(22.) A plane is inclined to the horizontal at an angle of $7^\circ 33'$. At a certain point on the plane a flag-pole subtends an angle $20^\circ 3'$, and at a point 50 ft. nearer the pole an angle of $40^\circ 35'$. Find the height of the pole.

(23.) The angle of elevation of an inaccessible tower, situated in a plane, is $53^\circ 19'$. At a point 227 ft. farther from the tower the angle of elevation is $22^\circ 41'$. Find the height of the tower.

(24.) A house stands on a hill which slopes $12^\circ 18'$ to the horizontal. 75 ft. from the house down the hill the house subtends an angle of $32^\circ 5'$. Find the height of the house.

(25.) From one bank of a river the angle of elevation of a tree on the opposite bank is $28^\circ 31'$. From a point 139.4 ft. farther away in a direct line its angle of elevation is $19^\circ 10'$. Find the width of the river.

(26.) From the foot of a hill in a plane the angle of elevation of the top of the hill is $21^\circ 7'$. After going directly away 211 ft. farther, the angle of elevation is $18^\circ 37'$. Find the height of the hill.

(27.) A monument at the top of a hill is 153.2 ft. high. At a point 321.4 ft. down the hill the monument subtends an angle of $11^\circ 13'$. Find the distance from this point to the top of the monument.

(28.) A building is situated on the top of a hill which is inclined $10^\circ 12'$ to the horizontal. At a certain distance up the hill the angle of elevation of the top of the building is $20^\circ 55'$, and 115.3 ft. farther down the hill the angle of elevation is $15^\circ 10'$. Find the height of the building.

(29.) A cloud, $C$, is observed from two points, $A$ and $B$, 2874 ft. apart, the line $AB$ being directly beneath the cloud. At $A$, the angle of elevation of the cloud is $77^\circ 19'$, and the angle $CAB$ is $51^\circ 18'$. The angle $ABC$ is found to be $60^\circ 45'$. Find the height of the cloud above $A$.

(30.) Two observers, $A$ and $B$, are on a straight road, 675.4 ft. apart, directly beneath a balloon, $C$. The angles $ABC$ and $BAC$ are $34^\circ 42'$ and $41^\circ 15'$ respectively. Find the distance of the balloon from the first observer.
MISCELLANEOUS EXERCISES

(31.) A man on the opposite side of a river from two objects, $A$ and $B$, wishes to obtain their distance apart. He measures the distance $CD = 337$ ft., and the angles $ACB = 29^\circ 33'$, $BCD = 38^\circ 52'$, $ADB = 54^\circ 10'$, and $ADC = 34^\circ 11'$. Find the distance $AB$.

(32.) A cliff is 327 ft. above the sea-level. From the top of the cliff the angles of depression of two ships are $15^\circ 11'$ and $13^\circ 13'$. From the bottom of the cliff the angle subtended by the ships are $122^\circ 39'$. How far are the ships apart?

(33.) A man standing on an inclined plane 112 ft. from the bottom observed the angle subtended by a building at the bottom to be $33^\circ 52'$. The inclination of the plane to the horizontal is $18^\circ 51'$. Find the height of the building.

(34) Two boats, $A$ and $B$, are 451.35 ft. apart. The angle of elevation of the top of a light-house, as observed from $A$, is $33^\circ 17'$. The base of the light-house, $C$, is level with the water; the angles $ABC$ and $CAB$ are $12^\circ 31'$ and $137^\circ 22'$ respectively. Find the height of the light-house.

(35.) From a window directly opposite the bottom of a steeple the angle of elevation of the top of the steeple is $23^\circ 21'$. From another window, 20 ft. vertically below the first, the angle of elevation is $39^\circ 3'$. Find the height of the steeple.

(36.) A dock is 1 mile from one end of a breakwater, and 1$\frac{1}{2}$ miles from the other end. At the dock the breakwater subtends an angle of $31^\circ 11'$. Find the length of the breakwater in feet.

(37.) A straight road ascending a hill is 1022 ft. long. The hill rises 1 ft. in every 4. A tower at the top of the hill subtends an angle of $7^\circ 19'$ at the bottom. Find the height of the tower.

(38.) A tower, 192 ft. high, rises vertically from one corner of a triangular yard. From its top the angles of depression of the other corners are $58^\circ 4'$ and $17^\circ 49'$. The side opposite the tower subtends from the top of the tower an angle of $75^\circ 15'$. Find the length of this side.

(39.) There are two columns left standing upright in a certain ruins; the one is 66 ft. above the plain, and the other 48. In a straight line between them stands an ancient statue, the head of which is 100 ft. from the summit of the higher, and 84 ft. from the top of the lower
column, the base of which measures just 74 ft. to the centre of the figure's base. Required the distance between the tops of the two columns.

(40.) Two sides of a triangle are in the ratio of 11 to 9, and the opposite angles have the ratio of 3 to 1. What are these angles?

(41.) The diagonals of a parallelogram are 12432 and 8413, and the angle between them is $78^\circ 44'$; find its area.

(42.) One side of a triangle is 1012.6 and the two adjacent angles are $52^\circ 21'$ and $57^\circ 32'$; find its area.

(43.) Two sides of a triangle are 218.12 and 123.72, and the included angle is $59^\circ 10'$; find its area.

(44.) Two angles of a triangle are $35^\circ 15'$ and $47^\circ 18'$, and the included side is 2104.7; find its area.

(45.) The three sides of a triangle are 1.2371, 1.4715, and 2.0721; find the area.

(46.) Two sides of a triangle are 168.12 and 179.21, and the included angle is $41^\circ 14'$; find its area.

(47.) The three sides of a triangle are 51 ft., 48.12 ft., and 32.2 ft.; find the area.

(48.) Two sides of a triangle are 111.18 and 121.21, and the included angle is $27^\circ 50'$; find its area.

(49.) The diagonals of a parallelogram are 37 and 51, and they form an angle of $65^\circ$; find its area.

(50.) If the diagonals of a quadrilateral are 34 and 56, and if they intersect at an angle of $67^\circ$, what is the area?
82. Let $O$ be the centre of a sphere of unit radius, and $ABC$ a right spherical triangle, right angled at $A$, formed by the intersection of the three planes $AOC$, $AOB$, and $BOC$ with the surface of the sphere. Suppose the planes $DAC''$ and $BEC'$ passed through the points $A$ and $B$ respectively, and perpendicular to the line $OC$. The plane angles $DC''A$ and $BC'E$ each measure the angle $C$ of the spherical triangle, and the sides of the spherical triangle $a, b, c$ have the same numerical measure as $BOC$, $AOC$, and $AOB$ respec-
tively, then, \( AD = \tan c \), \( BE = \sin c \), \( BC' = \sin a \), \( OC' = \cos a \), \( OC'' = \cos b \), \( OE = \cos c \), \( AC'' = \sin b \).

In the two similar triangles \( OEC' \) and \( OAC'' \),

\[
\frac{\cos c}{OA} = \frac{\cos c}{1} = \frac{\cos a}{\cos b}, \text{ or } \cos a = \cos b \cos c. \tag{1}
\]

In the triangle \( BC'E \),

\[
\sin C = \frac{BE}{BC'}, \text{ or } \sin C = \frac{\sin c}{\sin a}. \tag{2}
\]

In the triangle \( DAC'' \),

\[
\tan C = \frac{DA}{C''A}, \text{ or } \tan C = \frac{\tan c}{\sin b}. \tag{3}
\]

Combining formulas (2) and (3) with (1),

\[
\cos C = \frac{\tan b}{\tan a}. \tag{4}
\]

Again, if \( AB \) were made the base of the right spherical triangle \( ABC \), we should have

\[
\sin B = \frac{\sin b}{\sin a}. \tag{5}
\]

\[
\tan B = \frac{\tan b}{\sin c}. \tag{6}
\]

\[
\cos B = \frac{\tan c}{\tan a}. \tag{7}
\]

From the foregoing equations we may also obtain by combinations,

\[
\cos B = \sin C \cos b. \tag{8}
\]

\[
\cos C = \sin B \cos c. \tag{9}
\]

\[
\cos a = \cot B \cot C. \tag{10}
\]

**NAPIER'S RULES OF CIRCULAR PARTS**

83. The above ten formulas are sufficient to solve all cases of right spherical triangles. They may, however, be
expressed as two simple rules, called, after their inventor, Napier's rules.

The two sides adjacent to the right angle, the complement of the hypotenuse, and the complements of the oblique angles are called the **circular parts**.

The right angle is not one of the circular parts.

Thus there are *five* circular parts—namely, \( b, c, \text{comp}a, \text{comp}B, \text{comp}C \). Any one of the five parts may be called the *middle* part, then the two parts next to it are called *adjacent* parts, and the remaining two parts are called the *opposite* parts.

Thus if \( c \) is taken for the middle part, \( \text{comp}B \) and \( b \) are adjacent parts, and \( \text{comp}a \) and \( \text{comp}C \) are opposite parts.

The ten formulas may be written and grouped as follows:

**1st Group.**

\[
\begin{align*}
\sin \text{comp}C &= \tan \text{comp}a \tan b, \\
\sin \text{comp}B &= \tan \text{comp}a \tan c, \\
\sin c &= \tan \text{comp}B \tan \text{comp}C, \\
\sin b &= \tan \text{comp}C \tan c. \\
\end{align*}
\]

**2nd Group.**

\[
\begin{align*}
\sin \text{comp}a &= \cos b \cos c, \\
\sin b &= \cos \text{comp}a \cos \text{comp}B, \\
\sin c &= \cos \text{comp}a \cos \text{comp}C, \\
\sin \text{comp}B &= \cos \text{comp}C \cos b, \\
\sin \text{comp}C &= \cos \text{comp}B \cos c. \\
\end{align*}
\]

Napier’s rules may be stated:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*
84. In the right spherical triangles considered in this work, each side is taken less than a semicircumference, and each angle less than two right angles.

In the solution of the triangles, it is to be observed,

(1.) If the two sides about the right angle are both less or both greater than 90°, the hypotenuse is less than 90°; if one side is less and the other greater than 90°, the hypotenuse is greater than 90°.

(2.) An angle and the side opposite are either both less or both greater than 90°.

EXAMPLE

85. Given \(a = 63° 56', b = 40° 09',\) to find \(c, B,\) and \(C.\)

<table>
<thead>
<tr>
<th>To find (c.)</th>
<th>To find (B.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c) and (b) are the opposite parts. (\sin \text{comp} a = \cos b \cos c,) (\text{or} \cos a = \cos b \cos c.)</td>
<td>(b) is the middle part. (\text{comp} a) and (\text{comp} B) are the opposite parts. (\sin b = \cos \text{comp} a \cos \text{comp} B.) (\text{or} \sin b = \sin a \sin B.)</td>
</tr>
<tr>
<td>(\cos c = \frac{\cos a}{\cos b}.)</td>
<td>(\sin B = \frac{\sin b}{\sin a}.)</td>
</tr>
<tr>
<td>(\log \cos a = 9.64288) (\log \cos b = 0.11575) (\log \cos c = 9.75863) (c = 54° 59' 47'')</td>
<td>(\log \sin b = 9.80807) (\log \cos a = 0.04659) (\log \sin B = 9.85466) (B = 45° 41' 28'')</td>
</tr>
</tbody>
</table>

| Use the three parts originally required. \(\text{comp} C\) is the middle part. \(\text{comp} a\) and \(\text{comp} c\) are opposite parts. \(\sin \text{comp} C = \cos c \cos \text{comp} B.\) \(\text{or} \cos c = \cos c \sin B.\) | Use the three parts originally required. \(\text{comp} C\) is the middle part. \(\text{comp} B\) and \(c\) are opposite parts. \(\sin \text{comp} C = \cos c \cos \text{comp} B.\) \(\text{or} \cos c = \cos c \sin B.\) |
| \(\log \cot a = 9.68946\) \(\log \tan b = 9.92381\) \(9.61327\) \(C = 65° 45' 58''\) | \(\log \cos c = 9.75863\) \(\log \sin B = 9.85466\) \(\log \cos C = 9.61329\) \(C = 65° 45' 54''\) |
AMBIGUOUS CASE

86. When a side about the right angle and the angle opposite this side are given, there are two solutions, as illustrated by the following figure. Since the solution gives the values of each part in terms of the sine, the results are not only the values of \( a, b, B \), but \( 180^\circ - a, 180^\circ - b, 180^\circ - B \).

Given \( c = 26^\circ 4' \).
\( C = 36^\circ 0' \).

To find \( a, a', b, b' \) and \( B, B' \), using Napier's rules.

**To find \( B \) and \( B' \).**

To find \( B \) and \( B' \).
\[
\sin \text{comp} C = \cos \text{comp} B \cos c,
\]
or
\[
\cos C = \sin B \cos c,
\]
or
\[
\sin B = \frac{\cos C}{\cos c}.
\]

\[
\log \cos C = 9.90796
\]
\[
\colog \cos C = 0.04659
\]
\[
\log \sin B = 9.95455
\]
\[
B = 64^\circ 14' 30''
\]
\[
B' = 180^\circ - B = 115^\circ 45' 30''
\]

**To find \( a \) and \( a' \).**

To find \( a \) and \( a' \).
\[
\sin c = \cos \text{comp} a \cos \text{Comp} C,
\]
or
\[
\sin c = \sin a \sin C,
\]
or
\[
\sin a = \frac{\sin c}{\sin C}.
\]

\[
\log \sin c = 9.64288
\]
\[
\colog \sin c = 0.23078
\]
\[
\log \sin a = 9.87366
\]
\[
a = 48^\circ 22' 55'' -
\]
\[
a' = 180^\circ - a = 131^\circ 37' 5'' +
\]
(Discrepancy due to omitted decimals.)

**Check.**

To check.
\[
\sin b = \cos \text{comp} a \cos \text{Comp} B,
\]
or
\[
\sin b = \sin a \sin B.
\]
\[
\log \sin b = 9.82821
\]
\[
\log \sin b = 9.82821
\]
\[
b = 42^\circ 19' 17''
\]
\[
b' = 180^\circ - b = 137^\circ 40' 43''
\]
\[
b = 137^\circ 40' 39''
\]
QUADRANTAL TRIANGLES

87. Def.—A quadrantal triangle is a spherical triangle one side of which is a quadrant.

A quadrantal triangle may be solved by Napier's rules for right spherical triangles as follows:

By making use of the polar triangle where
\[ A = 180° - a' \quad a = 180° - A' \]
\[ B = 180° - b' \quad b = 180° - B' \]
\[ C = 180° - c' \quad c = 180° - C' \]

we see that the polar triangle of the quadrantal triangle is a right triangle which can be solved by Napier's rules. Whence we may at once derive the required parts of the quadrantal triangle.

EXAMPLE

Given \( A = 136° 4' \), \( B = 140° 0' \), \( a = 90° 0' \).

The corresponding parts of the polar triangle are
\( a' = 63° 56' \), \( b' = 40° 0' \), \( A' = 90° \).

By Napier's rules we find
\( B' = 45° 41' 28'' \), \( C' = 65° 45' 58'' \), \( c = 54° 59' 47'' \);

whence, by applying to these parts the rule of polar triangles, we obtain
\( b = 134° 18' 32'' \), \( c = 114° 14' 2'' \), \( C = 125° 0' 13'' \).

EXERCISES

88. (1.) In the right-angled spherical triangle \( ABC \), the side \( a = 63° 56' \), and the side \( b = 40° \). Required the other side, \( c \), and the angles \( B \) and \( C \).

(2.) In a right-angled triangle \( ABC \), the hypotenuse \( a = 91° 42' \), and the angle \( B = 95° 6' \). Required the remaining parts.

(3.) In the right-angled triangle \( ABC \), the side \( b = 26° 4' \), and the angle \( B = 36° \). Required the remaining parts.

(4.) In the right-angled spherical triangle \( ABC \), the side \( c = 54° 30' \), and the angle \( B = 44° 50' \). Required the remaining parts.

Why is not the result ambiguous in this case?
(5.) In the right-angled spherical triangle $ABC$, the side $b = 55^\circ 28'$, and the side $c = 63^\circ 15'$. Required the remaining parts.

(6.) In the right-angled spherical triangle $ABC$, the angle $B = 69^\circ 20'$, and the angle $C = 58^\circ 16'$. Required the remaining parts.

(7.) In the spherical triangle $ABC$, the side $a = 90^\circ$, the angle $C = 42^\circ 10'$, and the angle $A = 115^\circ 20'$. Required the remaining parts.

*Hint.*—The angle $A$ of the polar triangle is a right angle.

(8.) In the spherical triangle $ABC$, the side $b = 90^\circ$, the angle $C = 69^\circ 13' 46''$, and the angle $A = 72^\circ 12' 4''$. Required the remaining parts.

(9.) In the right-angled spherical triangle $ABC$, the angle $C = 23^\circ 27' 42''$, and the side $b = 10^\circ 39' 40''$. Required the angle $B$ and the sides $a$ and $c$.

(10.) In the right spherical triangle $ABC$, the angle $B = 47^\circ 54' 20''$, and the angle $C = 61^\circ 50' 29''$. Required the sides.
CHAPTER IX

OBLIQUE-ANGLED TRIANGLES

89. Let $O$ be the centre of a sphere of unit radius, and $ABC$ an oblique-angled spherical triangle formed by the three planes $AOB$, $BOC$, and $AOC$. Suppose the plane $AED$ passed through the point $A$ perpendicular to $AO$, intersecting the planes $AOB$, $BOC$, and $AOC$, in $AE$, $ED$, and $AD$ respectively. Then $AD = \tan b$, $AE = \tan c$, $OD = \sec b$, $OE = \sec c$.

In the triangle $EOD$,

$$ED^2 = \sec^2 b + \sec^2 c - 2 \sec b \sec c \cos a.$$  

In the triangle $AED$,

$$ED^2 = \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A.$$  

Subtracting these two equations and remembering that $\sec^2 b - \tan^2 b = 1$, we have

$$0 = 2 - 2 \sec b \sec c \cos a + 2 \tan b \tan c \cos A.$$  

Reducing, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$  

(i)
If we make \( b \) and \( c \) in turn the base of the triangle, we obtain in a similar way,

\[
\cos b = \cos c \cos a + \sin c \sin a \cos B,
\]

and

\[
\cos c = \cos a \cos b + \sin a \sin b \cos C.
\]

Remark.—In this group of formulas the second may be obtained from the first, and the third from the second, by advancing one letter in the cycle as shown in the figure; thus, writing \( b \) for \( a \), \( c \) for \( b \), \( a \) for \( c \), \( B \) for \( A \), \( C \) for \( B \), and \( A \) for \( C \). The same principle will apply in all the formulas of Oblique-Angled Spherical Triangles, and only the first one of each group will be given in the text.

**90.** By making use of the polar triangle where

\[
\begin{align*}
a &= 180^\circ - A' \\
b &= 180^\circ - B' \\
c &= 180^\circ - C'
\end{align*}
\]

we may obtain a second group of formulas.

Substituting these values of \( a, b, c, \) and \( A \) in (1), and remembering that \( \cos (180^\circ - A) = -\cos A \) and \( \sin (180^\circ - A) = \sin A \), we have

\[
\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a'.
\]

Since this is true for any triangle, we may omit the accents and write,

\[
\cos A = -\cos B \cos C + \sin B \sin C \cos a.
\]  

**FORMULAS FOR LOGARITHMIC COMPUTATION**

**91.** Formula (1), \( \cos a = \cos b \cos c + \sin b \sin c \cos A \),

gives

\[
\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.
\]

By § 36,

\[
\cos A = 1 - 2 \sin^2 \frac{1}{2} A
\]

Whence

\[
1 - 2 \sin^2 \frac{1}{2} A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},
\]

or

\[
\sin^2 \frac{1}{2} A = \frac{\cos b \cos c + \sin b \sin c - \cos a}{2 \sin b \sin c}.
\]
Let \( \frac{a+b+c}{2} = s \), then \( \frac{a+b-c}{2} = s-c \), and \( \frac{a-b+c}{2} = s-b \),

we have

\[
\sin \frac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}.
\]

Since, also, \( \cos A = 2 \cos^2 \frac{1}{2} A - 1 \),

we have, similarly,

\[
\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}.
\]

Hence

\[
\tan \frac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}. \tag{I}
\]

By a like process, formula (2) reduces to

\[
\tan \frac{1}{2} a = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}. \tag{II}
\]

92. If, in formula I, we advance one letter, we have

\[
\tan \frac{1}{2} B = \sqrt{\frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}}.
\]

And dividing \( \tan \frac{1}{2} A \) by \( \tan \frac{1}{2} B \), and reducing, we obtain

\[
\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \frac{\sin (s-b)}{\sin (s-a)}.
\]

By composition and division,

\[
\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)}.
\]

By \( \text{§} \text{§} 30, 38 \), this becomes

\[
\frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a-b)}. \tag{III}
\]
Multiplying \( \tan \frac{1}{2} A \) by \( \tan \frac{1}{2} B \), and reducing, we obtain

\[
\tan \frac{1}{2} A \tan \frac{1}{2} B \frac{1}{1} = \frac{\sin (s-c)}{\sin s}.
\]

By division and composition, and by \( \text{§} \text{§} 30, 38 \), this becomes

\[
\frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a+b)}.
\]  

(IV)

Proceeding in a similar way with formula II, we obtain

\[
\frac{\sin \frac{1}{2} (a+b)}{\sin \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A-B)}.
\]

(V)

And

\[
\frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} (a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A+B)}.
\]

(VI)

93. In the spherical triangle \( ABC \), suppose \( CD \) drawn perpendicularly to \( AB \), then, by the formulas for right spherical triangles,

\[
\begin{align*}
\sin p &= \sin b \sin A. \\
\sin p &= \sin a \sin B. \\
\text{Whence} \quad \sin a \sin B &= \sin b \sin A,
\end{align*}
\]

or

\[
\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}. 
\]

(VII)

Remark.—If \( (A+B) > 180^\circ \), then \( (a+b) > 180^\circ \), and if \( (A+B) < 180^\circ \), then \( (a+b) < 180^\circ \).
94. All cases of oblique-angled triangles may be solved by applying one or more of the formulas I, II, III, IV, V, VI, VII, as shown in the following cases.

CASES

(1.) Given three sides, to find the angles.

*Apply formula I.*  *Check: apply V or VI.*

(2.) Given three angles, to find the sides.

*Apply formula II.*  *Check: apply III or IV.*

(3.) Given two sides and the included angle.

*Apply V and VI, and VII.*  *Check: apply III or IV.*

(4.) Given two angles and included side.

*Apply III and IV, and VII.*  *Check: apply V or VI.*

(5.) Given two angles and an opposite side.

*Apply VII, V, and III.*  *Check: apply IV.*

(6.) Given two sides and an opposite angle.

*Apply VII, V, and IV.*  *Check: apply III.*

EXAMPLE—CASE (1)

95. Given $a = 81^\circ 10'$  $b = 60^\circ 20'$  $c = 112^\circ 25'$

To find $A$, $B$, and $C$.

<table>
<thead>
<tr>
<th>$a = 81^\circ 10'$</th>
<th>$b = 60^\circ 20'$</th>
<th>$c = 112^\circ 25'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2s = 253^\circ 55'$</td>
<td>$s = 126^\circ 57' 30''$</td>
<td></td>
</tr>
<tr>
<td>$s - a = 45^\circ 47' 30''$</td>
<td>$s - b = 66^\circ 37' 30''$</td>
<td></td>
</tr>
<tr>
<td>$s - c = 14^\circ 32' 30''$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To find $A$.

$$\tan \frac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c) \sin s \sin (s - a)}{\sin s \sin (s - a)}}$$

<table>
<thead>
<tr>
<th>log sin $(s - b)$</th>
<th>log sin $(s - c)$</th>
<th>log sin $s$</th>
<th>log sin $(s - a)$</th>
</tr>
</thead>
</table>

$$\frac{1}{2} A = 32^\circ 23' 19''$$

$A = 64^\circ 46' 38''$
To find $B$.

\[
\tan \frac{1}{2} B = \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin s \sin(s-b)}}.
\]

<table>
<thead>
<tr>
<th>\log \sin(s-a)</th>
<th>\log \sin(s-c)</th>
<th>\log \sin s</th>
<th>\log \sin(s-b)</th>
<th>\log \tan \frac{1}{2} B</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.85540</td>
<td>9.39982</td>
<td>0.09741</td>
<td>0.03719</td>
<td>9.69491</td>
</tr>
</tbody>
</table>

$B = 26^\circ 21' 6''$

$B = 52^\circ 42' 12''$

Check.

Formula V, $\cot \frac{1}{2} C = \frac{\tan \frac{1}{2} (A-B) \sin \frac{1}{2} (a+b)}{\sin \frac{1}{2} (a-b)}$

\[
A = 64^\circ 46' 38''
\]

\[
B = 52^\circ 42' 12''
\]

\[
A - B = 12^\circ 4' 26''
\]

\[
\frac{1}{2} (A-B) = 6^\circ 2' 13''
\]

\[a = 81^\circ 10'
\]

\[b = 60^\circ 20'
\]

\[a+b = 141^\circ 30'; \quad \frac{1}{2} (a+b) = 70^\circ 45',
\]

\[a-b = 20^\circ 50'; \quad \frac{1}{2} (a-b) = 10^\circ 25',
\]

\[\log \tan \frac{1}{2} (A-B) = 9.02430
\]

\[\log \sin \frac{1}{2} (a+b) = 9.97501
\]

\[\log \sin \frac{1}{2} (a-b) = 0.74279
\]

\[\cot \frac{1}{2} C = 9.74210
\]

\[\frac{1}{2} C = 61^\circ 5' 32''
\]

\[C = 122^\circ 11' 4''
\]

EXAMPLE—CASE (3)

96. Given $a = 78^\circ 15'$

\[b = 56^\circ 20'
\]

\[C = 120^\circ
\]

To find $A$, $B$, and $c$.

\[
\frac{1}{2} (a+b) = 67^\circ 17' 30''
\]

\[
\frac{1}{2} (a-b) = 10^\circ 57' 30''
\]

\[\frac{1}{2} C = 60^\circ
\]

To find $\frac{1}{2} (A+B)$.

Formula VI may be written

\[
\tan \frac{1}{2} (A+B) = \frac{\cos \frac{1}{2} (a-b) \cot \frac{1}{2} C}{\cos \frac{1}{2} (a+b)}.
\]

<table>
<thead>
<tr>
<th>\log \sin \frac{1}{2} (a+b)</th>
<th>\log \sin \frac{1}{2} (a-b)</th>
<th>\log \cot \frac{1}{2} C</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.99201</td>
<td>9.976144</td>
<td>9.76144</td>
</tr>
</tbody>
</table>

To find $\frac{1}{2} (A-B)$.

Formula V may be written

\[
\tan \frac{1}{2} (A-B) = \frac{\sin \frac{1}{2} (a-b) \cot \frac{1}{2} C}{\sin \frac{1}{2} (a+b)}.
\]

<table>
<thead>
<tr>
<th>\log \sin \frac{1}{2} (a+b)</th>
<th>\log \sin \frac{1}{2} (a-b)</th>
<th>\log \cot \frac{1}{2} C</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.96498</td>
<td>9.58663</td>
<td>9.76144</td>
</tr>
</tbody>
</table>

To find $\frac{1}{2} (A-B)$.

Formula V may be written

\[
\tan \frac{1}{2} (A-B) = \frac{\sin \frac{1}{2} (a-b) \cot \frac{1}{2} C}{\sin \frac{1}{2} (a+b)}.
\]

<table>
<thead>
<tr>
<th>\log \sin \frac{1}{2} (a+b)</th>
<th>\log \sin \frac{1}{2} (a-b)</th>
<th>\log \cot \frac{1}{2} C</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.58663</td>
<td>9.27897</td>
<td>9.76144</td>
</tr>
</tbody>
</table>
To find $c$.

From Formula VII, $\sin c = \frac{\sin b \sin C}{\sin B}$.

<table>
<thead>
<tr>
<th>Logarithms</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log \sin b$</td>
<td>9.92027</td>
</tr>
<tr>
<td>$\log \sin C$</td>
<td>9.93753</td>
</tr>
<tr>
<td>$\colog \sin B$</td>
<td>0.12249</td>
</tr>
<tr>
<td>$\log \sin c$</td>
<td>9.98029</td>
</tr>
</tbody>
</table>

$c = 107^\circ 8'$

Check.

Formula III may be written

$$\tan \frac{1}{2} c = \frac{\sin \frac{1}{2}(A + B) \tan \frac{1}{2}(a - b)}{\sin \frac{1}{2}(A - B)}.$$

<table>
<thead>
<tr>
<th>Logarithms</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log \sin \frac{1}{2}(A + B)$</td>
<td>9.91725</td>
</tr>
<tr>
<td>$\log \tan \frac{1}{2}(a - b)$</td>
<td>9.28696</td>
</tr>
<tr>
<td>$\colog \sin \frac{1}{2}(A - B)$</td>
<td>0.92762</td>
</tr>
<tr>
<td>$\log \tan \frac{1}{2} c$</td>
<td>10.13183</td>
</tr>
</tbody>
</table>

$$\frac{1}{2} c = 53^\circ 33' 56'' -$$

$$c = 107^\circ 7' 51'' -$$

(Discrepancy due to omitted decimals.)

**Ambiguous Cases**

97. (1.) Two sides and an angle opposite one of them are the given parts.

If the side opposite the given angle differs from $90^\circ$ more than the other given side, the given angle and the side opposite being either both less or both greater than $90^\circ$, there are two solutions.

(2.) Two angles and a side opposite one of them are the given parts.

If the angle opposite the given side differs from $90^\circ$ more than the other given angle, the given side and the angle opposite being either both less or both greater than $90^\circ$, there are two solutions.

**Remark.**—There is no solution if, in either of the formulas,

$$\sin B = \frac{\sin A \sin b}{\sin a}, \quad \sin a = \frac{\sin b \sin A}{\sin B},$$

the numerator of the fraction is greater than the denominator.
EXAMPLE—CASE (6)

98. Given \( a = 40^\circ 16' \) \( b = 47^\circ 44' \) \( A = 52^\circ 30' \).

To find \( B \), \( B' \), \( C \), \( C' \), and \( c \), \( c' \).

To find \( B \) and \( B' \).

Formula VII may be written

\[
\sin B = \frac{\sin A \sin b}{\sin a}
\]

\[
\log \sin A = 9.89947
\]

\[
\log \sin b = 9.86924
\]

\[
\colog \sin a = 0.18953
\]

\[
\log \sin B = 9.95824
\]

\[
B = 65^\circ 16' 30''
\]

\[
B' = 114^\circ 43' 30''
\]

To find \( c \).

Formula IV may be written

\[
\tan \frac{1}{2} c = \cos \frac{1}{2} (A + B) \tan \frac{1}{2} (a + b)
\]

\[
\log \cos \frac{1}{2} (A + B) = 9.71326
\]

\[
\log \tan \frac{1}{2} (a + b) = 9.98484
\]

\[
\colog \cos \frac{1}{2} (A - B) = 0.00270
\]

\[
\log \tan \frac{1}{2} c = 9.70080
\]

\[
\frac{1}{2} c = 26^\circ 39' 42''
\]

\[
c = 53^\circ 19' 24''
\]

To find \( c' \).

\[
\log \cos \frac{1}{2} (A + B') = 9.04631
\]

\[
\log \tan \frac{1}{2} (a + b) = 9.98484
\]

\[
\colog \cos \frac{1}{2} (A - B') = 0.06745
\]

\[
\log \tan \frac{1}{2} c' = 9.09860
\]

\[
\frac{1}{2} c' = 7^\circ 9' 9''
\]

\[
c' = 14^\circ 18' 18''
\]

To find \( C \).

Formula V may be written

\[
\cot \frac{1}{2} C = \frac{\sin \frac{1}{2} (a + b) \tan \frac{1}{2} (A - B)}{\sin \frac{1}{2} (a - b)}
\]

\[
\log \sin \frac{1}{2} (a + b) = 9.84177
\]

\[
\log \tan \frac{1}{2} (A - B) = 9.04901 \text{ n}
\]

\[
\colog \sin \frac{1}{2} (a - b) = 1.18633 \text{ n}
\]

\[
\log \cot \frac{1}{2} C = 10.07711
\]

\[
\frac{1}{2} C = 39^\circ 56' 24''
\]

\[
C = 79^\circ 52' 48''
\]

To find \( C' \).

\[
\log \sin \frac{1}{2} (a + b) = 9.84177
\]

\[
\log \tan \frac{1}{2} (A - B') = 9.78153 \text{ n}
\]

\[
\colog \sin \frac{1}{2} (a - b) = 1.18633 \text{ n}
\]

\[
\log \cot \frac{1}{2} C' = 10.80963
\]

\[
\frac{1}{2} C' = 8^\circ 48' 41''
\]

\[
C' = 17^\circ 37' 22''
\]

Check.

Formula III may be written

\[
\sin b = \frac{\sin B \sin c}{\sin C}
\]

\[
\log \sin B = 9.95824
\]

\[
\log \sin c = 9.90418
\]

\[
\colog \sin C = 0.00682
\]

\[
\log \sin b = 9.86924
\]

\[
b = 47^\circ 44'
\]

EXERCISES

99. (1.) In the spherical triangle \( ABC \), the side \( a = 124^\circ 53' \), the side \( b = 31^\circ 19' \), and the angle \( A = 16^\circ 26' \). Find the other parts.

(2.) In the oblique-angled spherical triangle \( ABC \), angle \( A = 128^\circ 45' \), angle \( C = 30^\circ 35' \), and the angle \( B = 68^\circ 50' \). Find the other parts.

*The letter "n" indicates that these quantities are negative.
(3.) In the spherical triangle \(ABC\), the side \(c = 78^\circ 15', b = 56^\circ 20'\), and \(A = 120^\circ\). Required the other parts.

(4.) In the spherical triangle \(ABC\), the angle \(A = 125^\circ 20'\), the angle \(C = 48^\circ 30'\), and the side \(b = 83^\circ 13'\). Required the remaining parts.

(5.) In the spherical triangle \(ABC\), the side \(c = 40^\circ 35', b = 39^\circ 10'\), and \(A = 71^\circ 15'\). Required the angles.

(6.) In the spherical triangle \(ABC\), the angle \(A = 109^\circ 55', B = 116^\circ 38', \) and \(C = 120^\circ 43'\). Required the sides.

(7.) In the spherical triangle \(ABC\), the angle \(A = 130^\circ 5' 22''\), the angle \(C = 36^\circ 45' 28''\), and the side \(b = 44^\circ 13' 45''\). Required the remaining parts.

(8.) In the spherical triangle \(ABC\), the angle \(A = 33^\circ 15' 7'', B = 31^\circ 34' 38'', \) and \(C = 161^\circ 25' 17''\). Required the sides.

(9.) In the spherical triangle \(ABC\), the side \(c = 112^\circ 22' 58'', b = 52^\circ 39' 4'', \) and \(a = 89^\circ 16' 53''\). Required the angles.

(10.) In the spherical triangle \(ABC\), the side \(c = 76^\circ 35' 36'', b = 50^\circ 10' 30'',\) and the angle \(A = 34^\circ 15' 3''\). Required the remaining parts.

**AREA OF THE SPHERICAL TRIANGLE**

100. It is proved in geometry that the area of a spherical triangle is equal to its spherical excess, that is, 
\[
\text{area} = (A + B + C - 2 \text{ rt. angles}) \times \text{area of the tri-rectangular triangle},
\]
where \(A\), \(B\), and \(C\) are the angles of the spherical triangle. Hence
\[
\frac{\text{area}}{\text{surface of sphere}} = \frac{A + B + C - 180^\circ}{\frac{720^\circ}{180^\circ}}.
\]
The surface of the sphere is \(4\pi R^2\), therefore
\[
\text{area} = \pi R^2 \left(\frac{A + B + C - 180^\circ}{180^\circ}\right)
\]
The following formula, called Lhuilier's theorem, simplifies the derivation of \((A + B + C - 180^\circ)\) where the three
sides of the spherical triangle are given; in it \( a, b, \) and \( c \) denote the sides of the triangle, and \( 2s=a+b+c \).

\[
\tan\left(\frac{A+B+C-180^\circ}{4}\right) = \sqrt{\tan \frac{1}{2} s \tan \frac{1}{2} (s-a) \tan \frac{1}{2} (s-b) \tan \frac{1}{2} (s-c)}.
\]

**EXERCISES**

(i.) The angles of a spherical triangle are, \( A=63^\circ, B=84^\circ 21', C=79^\circ \); the radius of the sphere is 10 in. What is the area of the triangle?

(2.) The sides of a spherical triangle are, \( a=6.47 \) in., \( b=8.39 \) in., \( c=9.43 \) in.; the radius of the sphere is 25 in. What is the area of the triangle?

(3.) In a spherical triangle, \( A=75^\circ 16', B=39^\circ 20', C=26 \) in.; the radius of the sphere is 14 in. Find the area of the triangle.

(4.) In a spherical triangle, \( a=4.41 \) miles, \( b=287 \) miles, \( C=38^\circ 21' \); the radius of the sphere is 3960 miles. Find the area of the triangle.
CHAPTER X

APPLICATIONS TO THE CELESTIAL AND TERRESTRIAL SPHERES

ASTRONOMICAL PROBLEMS

101. An observer at any place on the earth's surface finds himself seemingly at the centre of a sphere, one-half of which is the sky above him. This sphere is called the celestial sphere, and upon its surface appear all the heavenly bodies. The entire sphere seems to turn completely around once in 23 hours and 56 minutes, as on an axis. The imaginary axis is the axis of the earth indefinitely produced. The points in which it pierces the celestial sphere appear stationary, and are called the north and south poles of the heavens. The North Star (Polaris) marks very nearly (within 1° 16') the position of the north pole. As the observer travels towards the north he finds that the north pole of the heavens appears higher and higher up in the sky, and that its height above the horizon, measured in degrees, corresponds to the latitude of the place of observation.

The fixed stars and nebulæ preserve the same relative positions to each other. The sun, moon, planets, and comets change their positions with respect to the fixed stars continually, the sun appearing to move eastward among the stars about a degree a day, and the moon about thirteen times as far.
The zenith is the point on the celestial sphere directly overhead.

The horizon is the great circle everywhere 90° from the zenith.

The celestial equator is the great circle in which the plane of the earth’s equator if extended would cut the celestial sphere.

The ecliptic is the path on the celestial sphere described by the sun in its apparent eastward motion among the stars. The ecliptic is a great circle inclined to the plane of the equator at an angle of approximately 23\(\frac{1}{2}\)°.

The poles of the equator are the points where the axis of the earth if produced would pierce the celestial sphere, and are each 90° from the equator.

The poles of the ecliptic are each 90° from the ecliptic.

The equinoxes are the points where the celestial equator and ecliptic intersect; that which the sun crosses when coming north being called the vernal equinox, and that which it crosses when going south the autumnal equinox.

The declination of a heavenly body is its distance, measured in degrees, north or south of the celestial equator.

The right ascension of a heavenly body is the distance, measured in degrees eastward on the celestial equator, from the vernal equinox to the great circle passing through the poles of the equator and this body.

The celestial latitude of a heavenly body is the distance from the ecliptic measured in degrees on the great circle passing through the pole of the ecliptic and the body.

The celestial longitude of a heavenly body is the distance, measured in degrees eastward on the ecliptic, from
the vernal equinox to the great circle passing through the pole of the ecliptic and the body.

EXERCISES

(1.) The right ascension of a given star is $25° \ 35'$, and its declination is $+ (\text{north}) \ 63° \ 26'$. Assuming the angle between the celestial equator and the ecliptic to be $23° \ 27'$, find the celestial latitude and celestial longitude.

In this figure $AB$ is the celestial equator, $AC$ the ecliptic, $P$ the pole of the equator, $P'$ the pole of the ecliptic. $S$ is the position of the star, and the lines $SB$ and $SC$ are drawn through $P$ and $P'$ perpendicular to $AB$ and $AC$. $AB$ is the right ascension and $BS$ the declination of the star, while $AC$ is the longitude and $SC$ the latitude of the star.

In the spherical triangle $P'SS$, it will be seen that $P'S$ is the complement of the celestial latitude, $PS$ the complement of the declination, and $P'SS$ is $90°$ plus the right ascension. It is to be noted that $A$ is the vernal equinox.

(2.) The declination of the sun on December 21st is $- (\text{south}) \ 23° \ 27'$. At what time will the sun rise as seen from a place whose latitude is $41° \ 18'$ north?

The arc $ZS$ which is the distance from the zenith to the centre of the sun when the sun's upper rim is on the horizon is $90° \ 50'$. The $50'$ is made up of the sun's semi-diameter of $16'$, plus the correction for refraction of $34'$. 
(3.) The declination of the sun on December 21st is — (south) 23° 27'. At what time would the sun set as seen from a place in latitude 50° 35' north?

In these figures $P$ is the pole of the equator, $Z$ the zenith, $EQ$ the celestial equator. $AS$ is the declination of the sun, $ZS=90°\ 50'$, $PS=90°+\text{declination}$, $PZ=90°-$latitude. The problem is to find the angle $SPZ$. An angle of 15° at the pole corresponds to 1 hour of time.

GEOGRAPHICAL PROBLEMS

102. The **meridian** of a place is the great circle passing through the place and the poles of the earth.

The **latitude** of a place is the arc of the meridian of the place extending from the equator to the place.

Latitude is measured north and south of the equator from 0° to 90°.

The **longitude** of a place is the arc of the equator extending from the zero meridian to the meridian of the place. The meridian of the Greenwich Observatory is usually taken as the zero meridian.

Longitude is measured east or west from 0° to 180°.

The longitude of a place is also the angle between the zero **meridian** and the meridian of the place.
In the following problems one minute is taken equal to one geographical mile.

(1.) Required the distance in geographical miles between two places, D and E, on the earth's surface. The longitude of D is $60^\circ 15'$ E., and the latitude $20^\circ 10'$ N. The longitude of E is $115^\circ 20'$ E., and the latitude $37^\circ 20'$ N.

In this figure $AC$ represents the equator of the earth, $P$ the north pole, and $A$ the intersection of the meridian of Greenwich with the equator. $PB$ and $PC$ represent meridians drawn through $D$ and $E$ respectively. Then $AB$ is the longitude and $BD$ the latitude of $D$; $AC$ the longitude and $CE$ the latitude of $E$.

(2.) Required the distance from New York, latitude $40^\circ 43'$ N., longitude $74^\circ 0'$ W., to San Francisco, latitude $37^\circ 48'$ N., longitude $122^\circ 28'$ W., on the shortest route.

(3.) Required the distance from Sandy Hook, latitude $40^\circ 28'$ N., longitude $74^\circ 1'$ W., to Madeira, in latitude $32^\circ 28'$ N., longitude $16^\circ 55'$ W., on the shortest route.

(4.) Required the distance from San Francisco, latitude $37^\circ 48'$ N., longitude $122^\circ 28'$ W., to Batavia in Java, latitude $6^\circ 9'$ S., longitude $106^\circ 53'$ E., on the shortest route.

(5.) Required the distance from San Francisco, latitude $37^\circ 48'$ N., longitude $122^\circ 28'$ W., to Valparaiso, latitude $33^\circ 2'$ S., longitude $71^\circ 41'$ W., on the shortest route.
CHAPTER XI

GRAPHICAL SOLUTION OF A SPHERICAL TRIANGLE

103. The given parts of a spherical triangle may be laid off, and then the required parts may be measured, by making use of a globe fitted to a hemispherical cup.

The sides of the spherical triangle are arcs of great circles, and may be drawn on the globe with a pencil, using the rim of the cup, which is a great circle, as a ruler. The rim of the cup is graduated from $0^\circ$ to $180^\circ$ in both directions.

The angle of a spherical triangle may be measured on a great circle drawn on the sphere at a distance of $90^\circ$ from the vertex of the angle.*

Case I. Given the sides $a$, $b$, and $c$ of a spherical triangle, to determine the angles $A$, $B$, and $C$.

Place the globe in the cup, and draw upon it a line equal to the number of degrees in the side $c$, using the rim of the cup as a ruler. Mark the extremities of this line $A$ and $B$. With $A$ and $B$ as centres, and $b$ and $a$ respectively as radii, draw with the dividers two arcs intersecting at $C$ (Fig. 1). Then, placing the globe in the cup so that the points $A$ and $C$ shall rest on the rim, draw the line $AC = b$, and in the same way draw $BC = a$.

To measure the angle $A$ place the arc $AB$ in coincidence

* Slated globes, three inches in diameter, made of papier-maché, and held in metal hemispherical cups, are manufactured for the use of students of spherical trigonometry at a small cost.
with the rim of the cup, and make $A\bar{E}$ equal to $90^\circ$. Also make $AF$ in $AC$ produced equal to $90^\circ$. Then place the globe in the cup so that $E$ and $F$ shall be in the rim, and note the measure of the arc $EF$. This is the measure of the angle $A$. In the same way the angles $B$ and $C$ can be determined.

**Case II.** Given the angles $A$, $B$, and $C$, to find the sides $a$, $b$, and $c$.

Subtract $A$, $B$, and $C$ each from $180^\circ$, to obtain the sides $a'$, $b'$, and $c'$ of the polar triangle. Construct this polar triangle according to the method employed in Case I. Mark its vertices $A'$, $B'$, and $C'$ . With each of these vertices as a centre, and a radius equal to $90^\circ$, describe arcs with the dividers. The points of intersection of these arcs will be the vertices $A$, $B$, and $C$ of the given triangle. The sides of this triangle $a$, $b$, and $c$ can then be measured on the rim of the cup.
**Case III.** Given two sides, $b$ and $c$, and the included angle $A$, to find $B$, $C$, and $a$.

Lay off (Fig. 3) the line $AB$ equal to $c$, and mark the point $D$ in $AB$ produced, so that $AD$ equals $90^\circ$. With the dividers mark another point, $F$, at a distance of $90^\circ$ from $A$. Turn the globe in the cup till $D$ and $F$ are both in the rim, and make $DE$ equal to the number of degrees in the angle $A$. With $A$ and $E$ in the rim of the cup, draw the line $AC$ equal to the number of degrees in the side $b$. Join $C$ and $B$. The required parts of the triangle can then be measured.

**Case IV.** Given the angles $A$ and $B$ and the included side $c$, to find $a$, $b$, and $C$.

Lay off the line $AB$ equal to $c$. Then construct the given angles at $A$ and $B$, as in Case III., and extend their sides to intersect at $C$.

**Case V.** Given the sides $b$, $a$, and the angle $A$ opposite one of these sides, to find $c$, $B$, and $C$. (Ambiguous case.)
Lay off (Fig. 4) $AC$ equal to $b$, and construct the angle $A$ as in Case III. Take $c$ in the dividers as a radius, and with $C$ as a centre describe arcs cutting the other side of the triangle in $B$ and $B'$, and measure the remaining parts of the two triangles.

If the arc described with $C$ as a centre does not cut the other side of the triangle, there is no solution. If tangent, there is one solution.

**Case VI. Given the angles $A, B,$ and the side $a$ opposite one of the angles.**

Construct the polar triangle of the given triangle by Case V.; then construct the original triangle as in Case II., and measure the parts required.

The constructions given above include all cases of right and quadrantal triangles.
CHAPTER XII

RECAPITULATION OF FORMULAS

ELEMENTARY RELATIONS (§ 10)

\[ \tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \]
\[ \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \]
\[ \tan x \cot x = 1, \quad \sin^2 x + \cos^2 x = 1, \]
\[ 1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x. \]

RIGHT TRIANGLES (§§ 14 AND 27)

\[ \sin A = \frac{a}{c}, \quad \sin B = \frac{b}{c}, \]
\[ \cos A = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \]
\[ \tan A = \frac{a}{b}, \quad \tan B = \frac{b}{a}, \]
\[ \cot A = \frac{b}{a}, \quad \cot B = \frac{a}{b}, \]
\[ a^2 + b^2 = c^2, \]

where \( c \) = hypotenuse, \( a \) and \( b \) sides about the right angle; \( A \) and \( B \) the acute angles opposite \( a \) and \( b \).

FUNCTIONS OF TWO ANGLES (§§ 30–34)

\[ \sin (x + y) = \sin x \cos y + \cos x \sin y, \]
\[ \sin (x - y) = \sin x \cos y - \cos x \sin y, \]
\[ \cos (x + y) = \cos x \cos y - \sin x \sin y, \]
\[ \cos (x - y) = \cos x \cos y + \sin x \sin y. \]
\[ \tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \]
\[ \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}, \]
\[ \cot (x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}, \]
\[ \cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \]

**FUNCTIONS OF TWICE AN ANGLE (§ 36)**

\[ \sin 2x = 2 \sin x \cos x, \]
\[ \cos 2x = \cos^2 x - \sin^2 x, \]
\[ = 1 - 2 \sin^2 x, \]
\[ = 2 \cos^2 x - 1, \]
\[ \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \]
\[ \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \]

**FUNCTIONS OF HALF AN ANGLE (§ 37)**

\[ \sin \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{2}}, \]
\[ \cos \frac{1}{2} x = \pm \sqrt{\frac{1 + \cos x}{2}}, \]
\[ \tan \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}, \]
\[ \cot \frac{1}{2} x = \sqrt{\frac{1 + \cos x}{1 - \cos x}}. \]

**SUMS AND DIFFERENCES OF FUNCTIONS (§ 38)**

\[ \sin u + \sin v = 2 \sin \frac{1}{2} (u + v) \cos \frac{1}{2} (u - v), \]
\[ \sin u - \sin v = 2 \cos \frac{1}{2} (u + v) \sin \frac{1}{2} (u - v), \]
\[ \cos u + \cos v = 2 \cos \frac{1}{2} (u + v) \cos \frac{1}{2} (u - v), \]
\[ \cos u - \cos v = -2 \sin \frac{1}{2} (u + v) \sin \frac{1}{2} (u - v), \]
\[ \frac{\sin u + \sin v}{\sin u - \sin v} = \frac{\tan \frac{1}{2} (u + v)}{\tan \frac{1}{2} (u - v)}. \]
OB LIQUE TRIANGLES (§§ 42-45)

\[
\begin{align*}
\frac{a}{\sin A} &= \frac{b}{\sin B} ; \\
\frac{a}{\sin A} &= \frac{c}{\sin C} ; \\
\frac{b}{\sin B} &= \frac{c}{\sin C} .
\end{align*}
\]

\[
\begin{align*}
a - b &= \tan \frac{1}{2} (A - B) \\
a + b &= \tan \frac{1}{2} (A + B) ,
\end{align*}
\]

\[
\begin{align*}
a - c &= \tan \frac{1}{2} (A - C) \\
a + c &= \tan \frac{1}{2} (A + C) ,
\end{align*}
\]

\[
\begin{align*}
b - c &= \tan \frac{1}{2} (B - C) \\
b + c &= \tan \frac{1}{2} (B + C) .
\end{align*}
\]

\[
\begin{align*}
\tan \frac{1}{2} A &= \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} , \\
\tan \frac{1}{2} B &= \sqrt{\frac{(s - c)(s - a)}{s(s - b)}} , \\
\tan \frac{1}{2} C &= \sqrt{\frac{(s - a)(s - b)}{s(s - c)}} ,
\end{align*}
\]

where \( s = \frac{a + b + c}{2} \).

\[
\begin{align*}
\tan \frac{1}{2} A &= \frac{K}{s - a} , \\
\tan \frac{1}{2} B &= \frac{K}{s - b} , \\
\tan \frac{1}{2} C &= \frac{K}{s - c} ,
\end{align*}
\]

where \( K = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \).

AREA OF A TRIANGLE (§ 46)

\[
S = \frac{1}{2} ac \sin B . \quad S = \frac{1}{2} ba \sin C . \quad S = \frac{1}{2} cb \sin A .
\]

\[
S = \sqrt{s(s - a)(s - b)(s - c)} .
\]

LOGARITHMIC, COSINE, SINE, AND EXPONENTIAL SERIES

(§ 58)

\[
\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + , \text{ etc.}
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + , \text{ etc.}
\]
Recapitulation of Formulas

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +, \text{ etc.} \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} +, \text{ etc.} \]

DE MOIVRE'S THEOREM (§ 71)

\[ (\cos x + \sqrt{-1} \sin x)^n = \cos nx + \sqrt{-1} \sin nx. \]
\[ \sin nx = n \cos^{n-1} x \sin x - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} x \sin^3 x +, \text{ etc} \]
\[ \cos nx = n \cos^n x - \frac{n(n-1)}{2!} \cos^{n-2} x \sin^2 x +, \text{ etc} \]

Hyperbolic Functions (§ 75)

\[ \sinh x = \frac{e^x - e^{-x}}{2}, \]
\[ \cosh x = \frac{e^x + e^{-x}}{2}, \]
\[ e^{ix} = \cos x + i \sin x. \]
\[ \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \]
\[ \cos x = \frac{e^{ix} + e^{-ix}}{2}. \]
\[ \sin ix = i \left( \frac{e^x - e^{-x}}{2} \right) = i \sinh x, \]
\[ \cos ix = \frac{e^x + e^{-x}}{2} = \cosh x. \]

Spherical Triangles

Right and Quadrantal Triangles (§§ 83, 87)

Use Napier's rules.

Oblique Triangles (§§ 89–93)

\[ \cos A = \cos b \cos c + \sin b \sin c \cos A. \]
\[ \cos A = - \cos B \cos C + \sin B \sin C \cos a. \]
\[ \tan \frac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}. \]
\[ \tan \frac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}. \]

\[
\begin{align*}
\sin \frac{1}{2} (A + B) &= \tan \frac{1}{2} c, \\
\sin \frac{1}{2} (A - B) &= \tan \frac{1}{2} (a - b), \\
\cos \frac{1}{2} (A + B) &= \tan \frac{1}{2} (a + b), \\
\cos \frac{1}{2} (A - B) &= \tan \frac{1}{2} (A - B).
\end{align*}
\]

\[
\begin{align*}
\sin \frac{1}{2} (a + b) &= \cot \frac{1}{2} C, \\
\sin \frac{1}{2} (a - b) &= \tan \frac{1}{2} (A - B), \\
\cos \frac{1}{2} (a + b) &= \cot \frac{1}{2} C, \\
\cos \frac{1}{2} (a - b) &= \tan \frac{1}{2} (A + B).
\end{align*}
\]

\[
\begin{align*}
\sin a &= \sin b, \\
\sin A &= \sin B.
\end{align*}
\]

**AREA OF SPHERICAL TRIANGLES (§ 101)**

\[
\text{area} = \pi R^2 \left( \frac{A + B + C - 180^\circ}{180^\circ} \right)
\]

\[
\tan \left( \frac{A + B + C - 180^\circ}{4} \right) = \sqrt{\tan \frac{1}{2} s \tan \frac{1}{2} (s - a) \tan \frac{1}{2} (s - b) \tan \frac{1}{2} (s - c)}. \]
APPENDIX

RELATIONS OF THE PLANE, SPHERICAL, AND PSEUDOSPHERICAL TRIGONOMETRIES

We have up to the present considered the trigonometries which deal with figures on a plane or spherical surface. A characteristic feature of these two surfaces is that the curvature of the plane is zero, while that of the sphere is a positive constant $\rho$. If the radius of the sphere is increased indefinitely, its surface approaches the plane as a limit while its curvature $\rho$ approaches 0.

In works on absolute geometry it is shown that there exists a surface which has a constant negative curvature: it is called a pseudo-sphere, and the trigonometry upon it pseudospherical trigonometry.

We observe that as $\rho$ passes continuously from positive to negative values, we pass from the sphere through the plane to the pseudo-sphere. Thus the formulas of plane trigonometry are the limiting cases of those of either of the two other trigonometries.

In the treatment of spherical trigonometry the radius of the sphere has been taken as unity. If, however, the radius of the sphere is $r$, and $a$, $b$, and $c$ denote the lengths of the sides of the spherical triangle, the formulas are changed, in that $a$ is replaced by $\frac{a}{r}$, $b$ by $\frac{b}{r}$, and $c$ by $\frac{c}{r}$; thus,
APPENDIX

\[ \sin C = \frac{\sin c}{\sin a} \]

becomes

\[ \sin C = \frac{\sin \frac{c}{r}}{\sin \frac{a}{r}} \]

The formulas for pseudo-spherical trigonometry are the same as the formulas of spherical trigonometry, except that the hyperbolic functions of \( \frac{a}{r}, \frac{b}{r}, \) and \( \frac{c}{r} \) are substituted for the trigonometric.

Thus, corresponding to the above formula of spherical trigonometry, is the formula

\[ \sin C = \frac{\sinh \frac{c}{r}}{\sinh \frac{a}{r}} \]

of pseudo-spherical trigonometry.

The pseudo-sphere is generated by revolving the curve whose equation is

\[ y = r \log \frac{r + \sqrt{r^2 - x^2}}{x} - \sqrt{r^2 - x^2} \]

about its \( y \) axis. The radius of the base of the pseudo-sphere is \( r \).
Hence the formulas of plane trigonometry can be derived from the formulas of either spherical or pseudo-spherical trigonometry by expressing the functions in series and allowing \( r \) to increase without limit.

**Example.**—Show that if \( r \) is increased indefinitely the following corresponding formulas for the spherical and pseudo-spherical right triangle

\[
\cos \frac{a}{r} = \cos \frac{b}{r} \cos \frac{c}{r},
\]

\[
\cosh \frac{a}{r} = \cosh \frac{b}{r} \cosh \frac{c}{r^3}
\]

reduce to the corresponding formula for a plane right triangle; that is, to

\[
a^2 = b^2 + c^2.
\]

Substituting the series \( \cos \frac{a}{r} \), etc., in equation (1), we obtain

\[
\left(1 - \frac{1}{2!} \left(\frac{a}{r}\right)^2 + \cdots \right) = \left(1 - \frac{1}{2!} \left(\frac{b}{r}\right)^2 + \cdots \right) \left(1 - \frac{1}{2!} \left(\frac{c}{r}\right)^2 + \cdots \right),
\]

or

\[
1 - \frac{1}{2!} \frac{a^2}{r^2} + \frac{1}{4!} \frac{a^4}{r^4} + \cdots = 1 - \frac{1}{2!} \frac{b^2}{r^2} - \frac{1}{2!} \frac{c^2}{r^2} + \frac{1}{4!} \frac{b^4}{r^4} + \cdots
\]

Substituting in equation (2) the series for \( \cosh \frac{a}{r} \), etc., which we obtain from

\( \cosh x = \frac{e^x + e^{-x}}{2} \), we have

\[
I + \frac{1}{2!} \left(\frac{a}{r}\right)^2 + \cdots = \left(I + \frac{1}{2!} \left(\frac{b}{r}\right)^2 + \cdots \right) \left(I + \frac{1}{2!} \left(\frac{c}{r}\right)^2 + \cdots \right),
\]

or

\[
I + \frac{1}{2!} \frac{a^2}{r^2} + \frac{1}{4!} \frac{a^4}{r^4} + \cdots = I + \frac{1}{2!} \frac{b^2}{r^2} + \frac{1}{2!} \frac{c^2}{r^2} + \frac{1}{4!} \frac{b^4}{r^4} + \cdots
\]

Cancelling \( 1 \) in equations (4) and (5), multiplying by \( r^2 \), and, finally, allowing \( r \) to increase without limit, we get from either equation

\[
a^2 = b^2 + c^2.
\]

**EXERCISES**

Derive each of the following formulas of plane trigonometry from the corresponding formula of spherical trigonometry, and also from the corresponding formula of pseudo-spherical trigonometry.
Right triangles; \( A = \text{right angle} \).

\[(1.) \text{ Plane, } \sin C = \frac{c}{a}.
\]

Spherical,
\[
\sin C = \frac{\sin c}{\sin a}.
\]

Pseudo-spherical,
\[
\sin C = \frac{\sinh c}{\sinh a}.
\]

\[\text{Oblique Triangles.}\]

\[(2.) \text{ Plane, } a^2 = b^2 + c^2 - 2 bc \cos A.
\]

Spherical,
\[
\cos a = \cos b \cos c + \sin b \sin c \cos A.
\]

Pseudo-spherical,
\[
\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A.
\]

\[(3.) \text{ Plane, } S = \sqrt{s(s-a)(s-b)(s-c)}.
\]

Spherical,
\[
\tan \left( \frac{A+B+C-180^\circ}{4} \right) = \sqrt{\tan \frac{s}{r} \tan \frac{(s-a)}{r} \tan \frac{(s-b)}{r} \tan \frac{(s-c)}{r}}.
\]

Pseudo-spherical,
\[
\tanh \left( \frac{180^\circ - A+B+C}{4} \right) = \sqrt{\tanh \frac{s}{r} \tanh \frac{(s-a)}{r} \tanh \frac{(s-b)}{r} \tanh \frac{(s-c)}{r}}.
\]
ANSWERS TO EXERCISES

§ 4 (page 3).

(1.) 192° 51' 25 1/2".
Quadrant III.
(2.) 25°.
(3.) 287°, 647°.
(4.) Quadrant III.

§ 9 (page 9).
tan 1000° is negative.
cos 810° is 0.
sin 760° is positive.
cot — 70° is negative.
cos 550° is negative.
tan 560° is negative.
tan 300° is positive.
cot 1560° is negative.
sin 130° is positive.
cos 260° is negative.
tan 310° is negative.

§ 13 (page 11).
(3.) cos 30° = 1/2 \sqrt{3},
tan 30° = -1/3 \sqrt{3},
cot 30° = -\sqrt{3},
sec 30° = 2 \sqrt{3},
csc 30° = -2.
(4.) cos x = -\sqrt{2}/3,
tan x = 1/2 \sqrt{2},
cot x = 2 \sqrt{2},
sec x = -3/4 \sqrt{2},
csc x = -3.

(5.) cos y = 1/2,
tan y = -\sqrt{3}/3,
cot y = -\sqrt{3}/3,
sec y = 2 \sqrt{3},
csc y = 2.
(6.) sin 60° = 1/2 \sqrt{3},
tan 60° = \sqrt{3},
cot 60° = 1/3 \sqrt{3},
sec 60° = 2,
csc 60° = 2/3 \sqrt{3}.
(7.) cos 0° = 1, tan 0° = 0.
(8.) sin z = 1/3, cos z = 2/3,
cot z = 3/4, sec z = 5/3,
csc z = 5/4.
(9.) sin 45° = cos 45° = 1/2 \sqrt{2},
tan 45° = 1,
sec 45° = csc 45° = \sqrt{2}.
(10.) sin y = -1/2 \sqrt{3}, cos y = -2/3,
cot y = 2 \sqrt{3}, sec y = -2/3,
csc y = -3/2 \sqrt{3}.
(11.) sin 30° = 1/2, cos 30° = 1/2 \sqrt{3},
tan 30° = 1/3 \sqrt{3},
sec 30° = 2 \sqrt{3},
csc 30° = 2.
(12.) sin x = 1/3, cos x = -2/3.
(13.) \sqrt{3} + 1/6 \sqrt{5}.

§ 17 (page 14).
(1.) sin 70° = cos 20°,
cos 60° = sin 30°,
cos 89° 31' = sin 29°,
cot 47° = tan 43°,
\[
\begin{align*}
\tan 63^\circ &= \cot 27^\circ, \\
\sin 72^\circ \ 39' &= \cos 17^\circ \ 21'.
\end{align*}
\]

(2.) \(x = 30^\circ\).

(3.) \(x = 22^\circ \ 30'\).

(4.) \(x = 18^\circ\).

(5.) \(x = 15^\circ\).

\[\text{§ 25 (page 21).}\]

(1.) \(225^\circ\) and \(315^\circ\),

\(60^\circ\) and \(240^\circ\).

(2.) \(60^\circ, 120^\circ, 420^\circ, 480^\circ\).

(3.) \(\sin -30^\circ = -\frac{1}{2}\),

\(\cos -30^\circ = \frac{\sqrt{3}}{2}\),

\(\sin 765^\circ = \cos 765^\circ = \frac{1}{2} \sqrt{2}\),

\(\sin 120^\circ = \frac{\sqrt{3}}{2}\),

\(\cos 120^\circ = -\frac{1}{2}\),

\(\sin 210^\circ = -\frac{1}{2}\),

\(\cos 210^\circ = -\frac{\sqrt{3}}{2}\).

(4.) The functions of \(405^\circ\) are equal to the functions of \(45^\circ\),

\(\sin 600^\circ = -\frac{\sqrt{3}}{2}\),

\(\cos 600^\circ = -\frac{1}{2}\),

\(\tan 600^\circ = \sqrt{3}\),

\(\cot 600^\circ = \frac{1}{\sqrt{3}}\),

\(\sec 600^\circ = -2\),

\(\csc 600^\circ = -\frac{2}{\sqrt{3}}\).

The functions of \(1125^\circ\) are equal to the functions of \(45^\circ\),

\(\sin -45^\circ = -\frac{\sqrt{2}}{2}\),

\(\cos -45^\circ = \frac{\sqrt{2}}{2}\),

\(\tan -45^\circ = \cot -45^\circ = -1\),

\(\sec -45^\circ = \sqrt{2}\),

\(\csc -45^\circ = -\sqrt{2}\).

\(\sin 225^\circ = \cos 225^\circ = -\frac{1}{2} \sqrt{2}\),

\(\tan 225^\circ = \cot 225^\circ = 1\),

\(\sec 225^\circ = \csc 225^\circ = -\sqrt{2}\).

(5.) The functions of \(-120^\circ\) are the same as those of \(600^\circ\) given in (4),

\(\sin -225^\circ = -\frac{1}{2} \sqrt{2}\),

\(\cos -225^\circ = -\frac{1}{2} \sqrt{2}\),

\(\tan -225^\circ = \cot -225^\circ = -1\),

\(\sec -225^\circ = -\sqrt{2}\),

\(\csc -225^\circ = \sqrt{2}\),

\(\sin -420^\circ = -\frac{1}{2} \sqrt{3}\),

\(\cos -420^\circ = -\frac{1}{2} \sqrt{3}\),

\(\tan -420^\circ = -\sqrt{3}\),

\(\cot -420^\circ = -\frac{1}{2} \sqrt{3}\),

\(\sec -420^\circ = 2\),

\(\csc -420^\circ = -\frac{2}{\sqrt{3}}\).

The functions of \(3270^\circ\) are equal to the functions of \(30^\circ\),

(6.) \(\sin 233^\circ = -\cos 37^\circ\),

\(\cos 233^\circ = -\sin 37^\circ\),

\(\tan 233^\circ = \cot 37^\circ\),

\(\cot 233^\circ = \tan 37^\circ\),

\(\sec 233^\circ = -\csc 37^\circ\),

\(\csc 233^\circ = -\sec 37^\circ\).

\(\sin -197^\circ = \sin 17^\circ\),

\(\cos -197^\circ = -\cos 17^\circ\),

\(\tan -197^\circ = -\tan 17^\circ\),

\(\cot -197^\circ = -\cot 17^\circ\),

\(\sec -197^\circ = -\sec 17^\circ\),

\(\csc -197^\circ = \csc 17^\circ\).

\(\sin 894^\circ = \sin 6^\circ\),

\(\cos 894^\circ = -\cos 6^\circ\),

\(\tan 894^\circ = -\tan 6^\circ\),

\(\cot 894^\circ = -\cot 6^\circ\),

\(\sec 894^\circ = -\sec 6^\circ\),

\(\csc 894^\circ = \csc 6^\circ\).

(7.) \(\sin 267^\circ = -\sin 87^\circ\),

\(\tan -254^\circ = -\tan 74^\circ\),

\(\cos 950^\circ = -\cos 50^\circ\).

(8.) \(-0.28\).
(9.) \(2 \sin^2 x\).
(10.) \(\sec^2 x - 1\).

(11.) \(\sin (x - 90^\circ) = -\cos x\),
\(\cos (x - 90^\circ) = \sin x\),
\(\tan (x - 90^\circ) = -\cot x\),
\(\cot (x - 90^\circ) = -\tan x\),
\(\sec (x - 90^\circ) = \csc x\),
\(\csc (x - 90^\circ) = -\sec x\).

§ 28 (page 24).

(1.) \(a = 62.324\),
\(A = 32^\circ 52' 40''\).

(2.) \(b = 21.874\),
\(A = 39^\circ 45' 28''\),
\(B = 50^\circ 14' 32''\).

(3.) \(a = 300.95\),
\(b = 683.96\),
\(B = 66^\circ 15'\).

(4.) \(b = 26.608\),
\(c = 45.763\),
\(B = 35^\circ 33'\),
area = 495.34.

(5.) \(b = 3.9973\),
\(c = 4.1537\),
\(A = 15^\circ 46' 33''\),
area = 2.257.

(6.) \(b = 0.01729\).

(7.) \(a = 298.5\).

(8.) \(A = 39^\circ 42' 24''\).

(9.) \(c = 2346.7\).

(10.) \(B = 28^\circ 57' 8''\).

(11.) 444.16 ft.

(12.) 186.32 ft.

(13.) 34° 33' 44''.

(14.) 303.99 ft.

(15.) 238.33 ft.

(16.) 15 miles (about).

(17.) 79.079 ft.

(18.) 165.68 ft.

(19.) \(53^\circ 33'\).

(20.) 115.136 ft.

(21.) 76.355 ft.

(22.) \(B = 80^\circ 32''\),
\(A = C = 49^\circ 59' 44''\).

(23.) \(B = 53^\circ 16' 36''\),
\(b = 12.0518\) in.,
area = 72.392 sq. in.

(24.) \(b = 130.52\) in.,
area = 24246 sq. in.

(25.) 23.263 ft.

(26.) 17° 48''.

(27.) \(5.3546\) in.

(28.) \(108.4950\) sq. ft.

(29.) 17 ft., \(885\) sq. ft.

(30.) radius = \(24.882\) in.,
apothem = \(20.13\) in.,
area = \(1472\) sq. in.

(31.) \(12.861\).

(32.) \(1782.3\) sq. ft.

(33.) \(38168\) ft.

(34.) \(20.21\) ft.

(35.) \(2518.2\) ft.

§ 29 (page 28).

(1.) \(A = 22^\circ 58'\),
\(b = 7.07\),
\(c = 9.0046\).

(2.) \(b = 79.435\),
\(A = 45^\circ 27' 14''\),
\(C = 95^\circ 24' 46''\).

(3.) \(AB = 7.6745\),
\(AB' = 2.6435\),
\(B = 46^\circ 43' 50''\),
\(B' = 135^\circ 16' 10''\),
\(ACB = 105^\circ 53' 10''\),
\(ACB' = 19^\circ 20' 50''\).

(4.) \(A = 37^\circ 53'\),
\(B = 45^\circ 52' 25''\).
\( C = 98^\circ 14' 35'' \).

(5.) 902.94.

(6.) 1253.2 ft.

(7.) 357.224 ft.

(8.) \( A = 44^\circ 2' 9'' \),
\( B = 51^\circ 28' 11'' \),
\( C = 84^\circ 29' 40'' \),
area = 126100 sq. ft.

(9.) 407.89 ft.

(10.) \( B = 121^\circ 21' 16'' \),
\( C = 92^\circ 6' 38'' \),
\( D = 71^\circ 11' 6'' \).

(11.) \( BC = 5.672 \),
\( DC = 3.694 \).

\( \S\ 39 \) (page 37).

(5.) \( \sin(45^\circ - x) = \frac{\sqrt{2}}{2} (\cos x - \sin x) \),
\( \cos(45^\circ - x) = \frac{\sqrt{2}}{2} (\cos x + \sin x) \),
\( \sin(45^\circ + x) = \frac{\sqrt{2}}{2} (\cos x + \sin x) \),
\( \cos(45^\circ + x) = \frac{\sqrt{2}}{2} (\cos x - \sin x) \).

(6.) \( \tan 75^\circ = 2 + \sqrt{3} \),
\( \tan 15^\circ = 2 - \sqrt{3} \).

(14.) \( \sin \frac{1}{2} y = \sqrt{3 - \sqrt{3}} \),
\( \cos \frac{1}{2} y = \sqrt{3 + \sqrt{3}} \),
\( \tan \frac{1}{2} y = 3 - \sqrt{3} \).

(15.) \( \sin 2x = -\frac{24}{25} \),
\( \cos 2x = -\frac{7}{25} \).

(16.) \( \sin 22\frac{1}{2}^\circ = \frac{1}{2} \sqrt{2 - \sqrt{2}} \),
\( \cos 22\frac{1}{2}^\circ = \frac{1}{2} \sqrt{2 + \sqrt{2}} \),
\( \tan 22\frac{1}{2}^\circ = \sqrt{2 - 1} \),
\( \cot 22\frac{1}{2}^\circ = \sqrt{2 + 1} \),
\( \sec 22\frac{1}{2}^\circ = \sqrt{4 - 2\sqrt{2}} \),
\( \csc 22\frac{1}{2}^\circ = \sqrt{4 + 2\sqrt{2}} \).

(17.) \( \sqrt{\frac{51}{4}} \).

(18.) \( \sin 15^\circ = \frac{1}{2} \sqrt{2 - \sqrt{3}} \),
\( \cos 15^\circ = \frac{1}{2} \sqrt{2 + \sqrt{3}} \).
\[
\begin{align*}
tan 15^\circ &= 2 - \sqrt{3}, \\
cot 15^\circ &= 2 + \sqrt{3}, \\
sec 15^\circ &= 2 \sqrt{2 - \sqrt{3}}, \\
csc 15^\circ &= 2 \sqrt{2 + \sqrt{3}}. \\
\end{align*}
\]

(20.) \(sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x.\)

(21.) \(cos 5x = 5 \cos x - 20 \cos^3 x + 16 \cos^5 x.\)

(23.) The values of \(x < 360^\circ\) are, \(0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ.\)

(36.) \(tan x \tan y.\)

\[\text{§ 41 (page 40).}\]

(1.) \(sin^{-1} \frac{1}{2} \sqrt{2} = 45^\circ, 135^\circ, 225^\circ, 315^\circ,\) etc.,
\(cos^{-1} \frac{1}{2} = 60^\circ, 300^\circ,\) etc.,
\(tan^{-1} (-1) = 135^\circ, 315^\circ,\) etc.,
\(cos^{-1} 1 = 0^\circ, 360^\circ,\) etc.,
\(sin^{-1} (-\frac{1}{2}) = 210^\circ, 330^\circ,\) etc.

(2.) \(tan x = 3.\)

(3.) \(cos x = \pm \frac{1}{2}, tan x = \pm \frac{1}{3}.\)

(4.) \(sin (tan^{-1} \frac{1}{2} \sqrt{3}) = \pm \frac{1}{2}.\)

(5.) \(sin (cos^{-1} \frac{1}{2}) = \pm \frac{1}{2}.\)

(6.) \(cot (tan^{-1} \frac{1}{2}) = 17.\)

(7.) \(a = \frac{1}{2} \sqrt{3}.\)

(8.) \(45^\circ, 225^\circ.\)

(9.) \(x = 45^\circ, y = 180^\circ.\)

(10.) \(sin^{-1} a = 225^\circ.\)

\[\text{§ 48 (page 46).}\]

(1.) \(C = 121^\circ 33',\)
\(b = 2133.5,\)
\(c = 2477.8.\)

(2.) \(C = 55^\circ 41',\)
\(b = 534.05.\)

\[\begin{align*}
c &= 653.52. \\
(3.) C &= 45^\circ 34', \quad a = 1548.1, \quad b = 1293.7. \\
(4.) A &= 105^\circ 59', \quad a = 54.018, \quad c = 47.738. \\
(5.) B &= 68^\circ 58', \quad b = 5274.9, \quad c = 3730. \\
(6.) B &= 54^\circ 58', \quad a = 923.4, \quad c = 1187.7. \\
\end{align*}\]

\[\text{§ 49 (page 47).}\]

(1.) (1.) Two solutions.
(2.) One solution, a right triangle.
(3.) One solution.
(4.) Two solutions.

(2.) \(B = 16^\circ 57' 21'',\)
\(C = 15^\circ 50' 39'',\)
\(c = 0.32122.\)

(3.) \(c = 2.5719,\)
\(B = 13^\circ 15' 1'',\)
\(C = 14^\circ 13' 59''.\)

(4.) \(c = 93.59, \quad c' = 12.07,\)
\(B = 26^\circ 52' 7'', \quad B' = 133^\circ 7' 53',\)
\(C = 131^\circ 46' 53'', C' = 5^\circ 31' 7''.\)

(5.) No solution.

(6.) \(b = 1.0916, \quad b' = 0.36276,\)
\(A = 39^\circ 37' 16'', A' = 140^\circ 22' 44'',\)
\(B = 117^\circ 50' 44'', B' = 17^\circ 5' 16''.\)

\[\text{§ 50 (page 48).}\]

(1.) \(a = 0.0971,\)
\(B = 90^\circ 35' 36'',\)
\(C = 48^\circ 9' 24''.\)
\(S = 0.0053261.\)
(2.)  \( c = 14.211, \) 
\( A = 76^\circ 20' \ 5'', \) 
\( B = 44^\circ 52' \ 55'', \) 
\( S = 80.962. \)

(3.)  \( b = 85.892, \) 
\( A = 67^\circ 21' \ 42'', \) 
\( C = 62^\circ 48' \ 18'', \) 
\( S = 3962.8. \)

(4.)  \( a = 0.6767, \) 
\( B = 15^\circ 9' \ 21'', \) 
\( C = 131^\circ 19' \ 39'', \) 
\( S = 0.08141. \)

(5.)  \( c = 72.87, \) 
\( A = 40^\circ 50' \ 32'', \) 
\( B = 11^\circ 2' \ 28'', \) 
\( S = 422.65. \)

§ 51 (page 49).

(1.)  \( A = 55^\circ 20' \ 42'', \) 
\( B = 106^\circ 35' \ 36'', \) 
\( C = 18^\circ 3' \ 42'', \) 
\( S = 267.92. \)

(2.)  \( A = 34^\circ 24' \ 26'', \) 
\( B = 73^\circ 14' \ 56'', \) 
\( C = 72^\circ 20' \ 36'', \) 
\( S = 3.6143. \)

(3.)  \( A = 52^\circ 20' \ 24'', \) 
\( B = 107^\circ 19' \ 14'', \) 
\( C = 20^\circ 20' \ 24'', \) 
\( S = 1437.5. \)

(4.)  \( A = 97^\circ 48', \) 
\( B = 18^\circ 21' \ 48'', \) 
\( C = 63^\circ 50' \ 12'', \) 
\( S = 193.13. \)

(5.)  \( A = 54^\circ 20' \ 16'', \) 
\( B = 70^\circ 27' \ 46'', \) 
\( C = 54^\circ 72', \) 
\( S = 6090. \)

(6.)  \( A = 35^\circ 59' \ 30''. \)

B = 48^\circ 44' \ 32'',
C = 95^\circ 15' \ 56'',
S = 0.60709.

§ 52 (page 50).

(1.)  1116.6 ft.
(2.)  3081.8 yards.
(3.)  638.34 ft.,
\( 14653 \) sq. ft.
(4.)  4.1 and 8.1.
(5.)  13.27 miles.
(6.)  6667 ft. One solution.
(7.)  121.97.
(8.)  44^\circ 2' \ 56''.
(9.)  32.151 sq. miles.
(11.)  54^\circ 29' \ 12''.
(12.)  \( a = 12296 \) ft.,
\( c = 13955 \) ft.
(13.)  294.77 ft.
(14.)  222.1 ft.
(16.)  4211.8 ft.
(17.)  72.613 miles.
(18.)  51.035 ft.
(19.)  0.85872 miles.
(20.)  2.98 miles.
(21.)  1331.2 ft.
(22.)  8.2 miles.
(23.)  187.39 ft.
(24.)  0.6011.
(25.)  4.8112 miles.
(26.)  60^\circ 51' \ 8''.
(27.)  37.365 ft.
(28.)  3.2103 miles.
(29.)  10.532 miles.
(30.)  851.22 yards.
(31.)  9.5722 miles.
(32.)  6.1271 miles.
(33.)  280.47 ft.
(34.)  1126.1 ft.
(35.) 4.8134 miles.
(36.) 2728.25 ft.

§ 53 (page 56).

(1.) \(30^\circ = 0.5236,\)
\(45^\circ = 0.7854,\)
\(60^\circ = 1.0472,\)
\(120^\circ = 2.0944,\)
\(135^\circ = 2.3562,\)
\(720^\circ = 12.5664,\)
\(990^\circ = 17.2788.\)

(2.) \(\frac{\pi}{8} = 22^\circ 30',\)
\(\frac{\pi}{10} = 18^\circ,\)
\(\frac{\pi}{6} = 28^\circ 38' 53'',\)
\(\frac{\pi}{4} = 100^\circ 16' 4''.\)

(3.) 1.35, 0.54.

§ 74 (page 73).

(1.) \(\sin 4x = 4 \cos^3 x \sin x - 4 \cos x \sin^3 x,\)
\(\cos 4x = \cos^4 x - 6 \cos^2 x \sin^2 x + \sin^4 x.\)

(2.) \(\sin 6x = 6 \cos^5 x \sin x - 20 \cos^3 x \sin^3 x + 6 \cos x \sin^5 x,\)
\(\cos 6x = \cos^6 x - 15 \cos^4 x \sin^2 x + 15 \cos^2 x \sin^6 x - \sin^6 x.\)

(3.) \(x_0 = 1,\) \(x_1 = \frac{1}{2} + i \frac{\sqrt{3}}{2},\)
\(x_2 = -\frac{1}{2} + i \frac{\sqrt{3}}{2},\) \(x_3 = -1,\)
\(x_4 = -\frac{1}{2} - i \frac{\sqrt{3}}{2},\)
\(x_5 = \frac{1}{2} - i \frac{\sqrt{3}}{2}.\)

(4.) \(x_0 = 1,\) \(x_1 = 0.3090 + i 0.9511,\)
\(x_2 = -0.8090 + i 0.5878,\)
\(x_3 = -0.8090 - i 0.5878,\)
\(x_4 = 0.3090 - i 0.9511.\)

§ 77 (page 78).

(23.) \(x = 30^\circ.\)
(24.) \(y = 30^\circ.\)
(25.) \(x = 0^\circ\) or \(45^\circ.\)
(26.) \(x = 60^\circ.\)
(27.) \(y = 45^\circ.\)
(28.) \(y = 45^\circ.\)
(29.) \(x = 45^\circ.\)
(30.) \(x = 30^\circ.\)
(31.) \(x = 60^\circ.\)
(32.) \(x = 30^\circ.\)
(33.) No angle < \(90^\circ.\)
(34.) \(x = 30^\circ.\)
(35.) \(\sin 92^\circ = \cos 2^\circ.\)
(36.) \(\cos 127^\circ = -\sin 37^\circ.\)
(37.) \(\tan 320^\circ = -\tan 40^\circ.\)
(38.) \(\cot 350^\circ = -\cot 10^\circ.\)
(39.) \(\sin 265^\circ = -\cos 5^\circ.\)
(40.) \(\tan 171^\circ = -\tan 9^\circ.\)

(41.) \(\cos x = -\frac{1}{2} \sqrt{\frac{3}{3}},\)
\(\tan x = -\frac{3}{3} \sqrt{33},\)
\(\cot x = -\frac{1}{2} \sqrt{33},\)
\(\sec x = -\frac{7}{3} \sqrt{33},\)
\(\csc x = \frac{7}{3}.\)

(42.) \(\sin x = -\frac{1}{2} \sqrt{\frac{5}{5}},\)
\(\tan x = \frac{1}{3} \sqrt{\frac{3}{3}},\)
\(\cot x = \frac{1}{3} \sqrt{55},\)
\(\sec x = -\frac{1}{3},\)
\(\csc x = -\frac{8}{3} \sqrt{\frac{3}{3}}.\)

(43.) \(\sin x = -\frac{1}{3} \sqrt{\frac{1}{13}},\)
\(\cos x = -\frac{1}{3} \sqrt{13},\)
\(\cot x = \frac{\sqrt{13}}{3}, \sec x = -\frac{1}{3} \sqrt{13}.\)
ANSWERS TO EXERCISES

\[
csc x = -\frac{1}{2} \sqrt{13}.
\]

\[
\begin{align*}
(44.) & \quad \sin x = -\frac{7}{4} \sqrt{74}, \\
& \quad \cos x = \frac{5}{4} \sqrt{74}, \\
& \quad \tan x = -\frac{7}{5}, \quad \sec x = \frac{1}{2} \sqrt{74}, \\
& \quad \csc x = -\frac{1}{2} \sqrt{74}.
\end{align*}
\]

(45.) Quadrant II or IV.
(46.) Quadrant I or II.
(47.) Quadrant III or IV.
(48.) Quadrant I or II.
(49.) \(x = 0^\circ, 120^\circ, 180^\circ, 240^\circ\).
(50.) \(x = 30^\circ, 135^\circ, 150^\circ, 315^\circ\).
(51.) \(x = 0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ\).
(57.) 0.
(58.) \(a\).
(59.) \(2(a - b)\).
(60.) \(\frac{1}{2}(a^2 - b^2)\).

§ 78 (page 80).

(1.) 306.32 ft.
(2.) 831.06 ft.
(3.) 53° 28' 14".
(4.) 49.39 ft.
(5.) 0.43498 mile.
(6.) 209.53 ft.
(7.) 7.3185 ft.
(8.) 37° 36' 30".
(9.) 109.28 ft.
(10.) 502.46 ft.
(11.) 6799.8 ft.
(12.) 219.05 ft.
(13.) 491.76 ft.
(14.) 50° 32' 44".
(15.) 49° 44' 38".
(16.) 34.063 ft.
(17.) 32.326 ft., 29° 6' 35".
(18.) 5.6569 miles an hour.
(19.) 56.295 ft.
(20.) 103.09 ft.
(21.) 71° 33' 54".
(22.) 858,100 miles.
(23.) 238,850 miles.
(24.) 2163.4 miles.
(25.) 90,824,000 miles.
(26.) 432.08 ft.
(27.) 60.191 ft.
(28.) 0.32149 mile.
(29.) 193.77 ft., or 1632.9 ft.

§ 79 (page 83).

(1.) 3.416 ft.
(2.) 3.7865 ft.
(3.) 20.45 ft.
(4.) 36.024 ft.
(5.) 8.6058 sq. ft.
(6.) 181.23 in.
(7.) 2.9943 ft.
(8.) 5.131 in.
(9.) 25.92 ft.
(10.) 92° 11' 24".
(11.) 1.2491.
(12.) 33° 12' 4".
(13.) 11248 ft.
(14.) 0.60965 mile.
(15.) 1.3764.
(16.) 1.9755.
(17.) 19.882.
(18.) 0.9397.
(19.) 6.4984.
(20.) 3.4641.
(22.) 68.978.
(23.) 15.25.

§ 80 (page 84).

(78.) \(x = 90^\circ, 120^\circ, 240^\circ, 270^\circ\).
(79.) \(x = 0^\circ, 20^\circ, 45^\circ, 90^\circ, 100^\circ, 135^\circ, 140^\circ, 180^\circ, 220^\circ, 225^\circ, 260^\circ, 270^\circ, 315^\circ, 340^\circ\).
(80.) $x = 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ, 270^\circ$.
(81.) $x = 0^\circ, 45^\circ, 120^\circ, 240^\circ, 225^\circ, 270^\circ$.
(82.) $x = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.
(83.) $x = 0^\circ, 90^\circ, 210^\circ, 330^\circ$.
(84.) $x = 240^\circ, 300^\circ$.
(85.) $x = 210^\circ, 330^\circ$.
(86.) $x = 0^\circ, 90^\circ$.
(87.) $x = 0^\circ, 180^\circ$.
(88.) $x = 0^\circ, 180^\circ$.
(89.) $x = 0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ$.
(90.) $x = 45^\circ, 135^\circ, 225^\circ, 315^\circ$.
(91.) $x = 30^\circ, 150^\circ, 210^\circ, 330^\circ$.

§ 81 (page 88).

(1.) 2145.1 ft.
(2.) 12.458 miles.
(3.) 1.1033 miles.
(4.) 1508.4 ft.
(5.) 1719.3 yards.
(6.) 1.2564 miles.
(7.) 1346.3 ft.
(8.) 387.1 yards.
(9.) 5.1083 miles.
(10.) 3791.8 ft.
(11.) 4.4152 ft.
(12.) 280° 57' 20''.
(13.) 115.27.
(14.) 44.358 ft.
(15.) 92.258 ft.
(16.) 101° 32' 16''.
(17.) 0.83732 mile.
(18.) 539.1 ft.
(19.) 1.239.
(20.) 152.31 and 238.3.
(21.) 67.110 ft.
(22.) 32.071 ft.
(23.) 137.78 ft.

(24.) 55.74 ft.
(25.) 247.52 ft.
(26.) 556.34 ft.
(27.) 455.12 ft.
(28.) 18.825 ft.
(29.) 2639.4 ft.
(30.) 396.54 ft.
(31.) 287.75 ft.
(32.) 2280.6 ft.
(33.) 64.62 ft.
(34.) 127.98 ft.
(35.) 45.183 ft.
(36.) 4365.2 ft.
(37.) 140.17 ft.
(38.) 610.45 ft.
(39.) 156.66 ft.
(40.) 41° 48' 39'' and 125° 25' 57''.
(41.) 51,288,000.
(42.) 364,183.
(43.) 11586.
(44.) 947460.
(45.) 0.89782.
(46.) 9929.3.
(47.) 751.62 sq. ft.
(48.) 3145.9.
(49.) 855.1.
(50.) 876.34.

§ 88 (page 98).

(1.) $c = 54^\circ 59' 47''$, $B = 45^\circ 41' 28''$, $C = 65^\circ 45' 58''$.
(2.) $C = 71^\circ 36' 47''$, $b = 95^\circ 22'$, $c = 71^\circ 32' 14''$.
(3.) $C = 64^\circ 14' 30''$, $C' = 115^\circ 45' 30''$, $a = 48^\circ 22' 55''$, $a' = 131^\circ 37' 5''$, $c = 42^\circ 19' 17''$. 
\[c = 137^\circ 40' 43''\].

(4.) \[C = 65^\circ 49' 54''\],
\[a = 63^\circ 10' 6''\],
\[b = 38^\circ 59' 12''\].

(5.) \[a = 75^\circ 13' 1''\],
\[B = 58^\circ 25' 46''\],
\[C = 67^\circ 27' 1''\].

(6.) \[a = 98^\circ 21' 22''\],
\[b = 109^\circ 50' 8''\],
\[c = 115^\circ 13' 4''\].

(7.) \[B = 32^\circ 26' 9''\],
\[a = 84^\circ 14' 32''\],
\[c = 51^\circ 6' 12''\].

(8.) \[a = 80^\circ 5' 8''\],
\[b = 70^\circ 10' 36''\],
\[c = 145^\circ 5' 2''\].

(9.) \[A = 70^\circ 39' 4''\],
\[B = 48^\circ 36' 3''\],
\[C = 119^\circ 15' 2''\].

(10.) \[a = 40^\circ 0' 12''\],
\[B = 42^\circ 15' 11''\],
\[C = 121^\circ 36' 19''\].

§ 100 (page 109).

(1.) 80.895 sq. in.
(2.) 26.869 sq. in.
(3.) 158.41 sq. in.
(4.) 39533 sq. miles.

§ 101 (page 112).

(1.) SC = 48^\circ 2' 43''\],
\[AC = 52^\circ 53' 9''\].

(2.) 7:24 A.M.
(3.) 4 P.M.

§ 102 (page 114).

(1.) 3029½ miles.
(2.) 2229.8 miles.
(3.) 2748.5 miles.
(4.) 7516.3 miles.
(5.) 5109 miles.

THE END
**ELEMENTS OF GEOMETRY**

By ANDREW W. PHILLIPS, Ph.D., and IRVING FISHER, Ph.D., Professors in Yale University

<table>
<thead>
<tr>
<th>Plane and Solid Geometry,</th>
<th>Elements of Geometry.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry of Space</td>
<td>Logarithms of Numbers</td>
</tr>
</tbody>
</table>

ALTHOUGH this book meets all the requirements in geometry for entrance to all the colleges in the country, it aims to present more than a mere minimum course. Among its chief characteristics are: Rigor of treatment, clearness of presentation both in the form of statement and in the diagrams, natural and symmetrical methods of proof, and richness and variety of original problems.

¶ The geometric, or space axioms, viz.; the straight line axiom, the parallel axiom, and the superposition axiom are separated from those that relate to magnitudes in general, and are emphasized as the foundation on which the whole geometric superstructure is built.

¶ The definitions are distributed through the book as they are needed, instead of being grouped in long lists many pages in advance of the propositions to which they apply. An alphabetical index is added for easy reference.

¶ The constructions in the Plane Geometry are also distributed, so that the student is taught how to make a figure at the same time that he is required to use it in demonstration.

¶ In the Geometry of Space the figures consist of half-tone engravings from the photographs of actual models constructed for use in the class rooms of Yale University. By the side of these models are skeleton diagrams for the student to copy.

¶ Attention is also called to the theorems of proportion, the theory of limits, the use of corollaries as exercises to supply the need of inventional geometry, and the introduction to modern geometry.

**AMERICAN BOOK COMPANY**
MORE extended and more comprehensive than Milne’s High School Algebra, this work not merely states the principles and laws of algebra, but establishes them by rigorous proofs. The student first makes proper inferences, then expresses the inferences briefly and accurately, and finally proves their truth by deductive reasoning. The definitions are very complete, and special applications and devices have been added. The examples are numerous and well graded, and the explanations which accompany the processes, giving a more intelligent insight into the various steps, constitute a valuable feature. The book meets the requirements in algebra for admission to all of the colleges.

ADVANCED ALGEBRA.

THIS book covers fully all college and scientific school entrance requirements in advanced algebra. While the earlier pages are identical with the author’s Academic Algebra, more than 160 pages of new matter have been added. Among the new subjects considered are: incommensurable numbers, mathematical induction, probability, simple continued fractions, the theory of numbers, determinants, convergency of series, exponential and logarithmic series, summation of series, and the theory of equations, including graphical representation of functions of one variable, and approximation to incommensurable roots. Over 5,000 unsolved exercises and problems are included in the book. The treatment is full, rigorous, and scientific.
MERRILL’S MECHANICS is intended for the upper classes in secondary schools, and for the two lower classes in college. Only a knowledge of elementary algebra, plane geometry, and plane trigonometry is required for a thorough comprehension of the work.

By presenting only the most important principles and methods, the book overcomes many of the difficulties now encountered by students in collegiate courses who take up the study of analytic mechanics, without previously having covered it in a more elementary form. It treats the subject without the use of the calculus, and consequently does not bewilder the beginner with much algebraic matter, which obscures the chief principles.

The book is written from the standpoint of the student in the manner that experience has proved to be the one most easily grasped. Therefore, beyond a constant endeavor to abide by the fundamental precepts of teaching, no one method of presentation has been used to the exclusion of others. The few necessary experiments are suggested and outlined, but a more complete laboratory course can easily be supplied by the instructor.

The explanation of each topic is followed by a few well-chosen examples to fix and apply the principles involved. A number of pages are devoted to the static treatment of force, with emphasis on the idea of action and reaction. Four-place tables of the natural trigonometric functions are included.
ELEME NTS OF GEOLOGY

By ELIOT BLACKWELDER, Associate Professor of Geology, University of Wisconsin, and HARLAN H. BARROWS, Associate Professor of General Geology and Geography, University of Chicago.

A n introductory course in geology, complete enough for college classes, yet simple enough for high school pupils. The text is explanatory, seldom merely descriptive, and the student gains a knowledge not only of the salient facts in the history of the earth, but also of the methods by which those facts have been determined. The style is simple and direct. Few technical terms are used. The book is exceedingly teachable.

The volume is divided into two parts, physical geology and historical geology. It differs more or less from its predecessors in the emphasis on different topics and in the arrangement of its material. Factors of minor importance in the development of the earth, such as earthquakes, volcanoes, and geysers, are treated much more briefly than is customary. This has given space for the extended discussion of matters of greater significance. For the first time an adequate discussion of the leading modern conceptions concerning the origin and early development of the earth is presented in an elementary textbook.

The illustrations and maps, which are unusually numerous, really illustrate the text and are referred to definitely in the discussion. They are admirably adapted to serve as the basis for classroom discussion and quizzes, and as such constitute one of the most important features of the book. The questions at the end of the chapters are distinctive in that the answers are in general not to be found in the text. They may, however, be reasoned out by the student, provided he has read the text with understanding.

AMERICAN BOOK COMPANY
MAYNE & HATCH'S HIGH SCHOOL AGRICULTURE

By D. D. MAYNE, Principal of School of Agriculture and Professor of Agricultural Pedagogics, University of Minnesota; and K. L. HATCH, Professor of Agricultural Education, University of Wisconsin.

THIS course has a double value for pupils in the first years of the high school. On the one hand, it puts the study of agriculture on a serious basis and teaches the young beginner how he can carry on the work of a farm most profitably. On the other hand, it affords an interesting introduction to all the natural sciences, enabling the student to master certain definite principles of chemistry, botany, and zoölogy, and to understand their applications. A few experiments are included, which may be performed by the student or by the teacher before the class. But the subject is not made ultrascientific, forcing the student through the long process of laboratory method to rediscover what scientists have fully established.

The topics are taken up in the text in their logical order. The treatment begins with an elementary agricultural chemistry, in which are discussed the elements that are of chief importance in plant and animal life. Following in turn are sections on soils and fertilizers; agricultural botany; economic plants, including field and forage crops, fruits and vegetables; plant diseases; insect enemies; animal husbandry; and farm management.

The chapter on plant diseases, by Dr. E. M. Freeman, Professor of Botany and Vegetable Pathology, College of Agriculture, University of Minnesota, describes the various fungus growths that injure crops, and suggests methods of fighting them. The section on farm management treats farming from the modern standpoint as a business proposition.
A BRIEF COURSE IN
GENERAL PHYSICS

By GEORGE A. HOADLEY, A.M., C.E.,
Professor of Physics, Swarthmore College

A COURSE, containing a reasonable amount of work for
an academic year, and covering the entrance require-
ments of all of the colleges. It is made up of a reliable
text, class demonstrations of stated laws, practical questions
and problems on the application of these laws, and laboratory
experiments to be performed by the students.

The text, which is accurate and systematically arranged,
presents the essential facts and phenomena of physics clearly
and concisely. While no division receives undue prominence,
stress is laid on the mechanical principles which underlie the
whole, the curve, electrical measurements, induced currents,
the dynamo, and commercial applications of electricity.

The illustrative experiments and laboratory work, intro-
duced at intervals throughout the text, are unusually numerous,
and can be performed with comparatively simple apparatus.
Additional laboratory work is included in the appendix, to-
gether with formulas and tables.

HOADLEY'S PRACTICAL MEASUREMENTS IN
MAGNETISM AND ELECTRICITY.

THIS book, which treats of the fundamental measurements in elec-
tricity as applied to the requirements of modern life, furnishes a satis-
factory introduction to a course in electrical engineering for secondary
and manual training schools, as well as for colleges. Nearly 100 experiments
are provided, accompanied by suggestive directions. Each experiment is
followed by a simple discussion of the principles involved, and, in some
cases, by a statement of well-known results.

AMERICAN BOOK COMPANY

(159)
ELEMENTS OF
POLITICAL ECONOMY
By J. LAURENCE LAUGHLIN, Ph.D., Head Pro-
fessor of Political Economy, University of Chicago

In the present edition the entire work is thoroughly revised
and as regards both theory and practical data is entirely
in accord with the times. The treatment is sufficiently
plain for even high school students.

The book is in two parts: Part I, pertaining to the prin-
ciples of political economy and containing chapters on the
many phases of production, exchange, and distribution; and
Part II, treating of such important topics as socialism, taxation,
the national debt, free trade and protection, bimetallism, United
States notes, banking, the national banking system, the labor
problem, and cooperation.

The work is equally suitable for a short or a long
course, and contains many valuable practical exercises which
are intended to stimulate thought on the part of the stu-
dent. A large bibliography, footnotes, and references are
included.

Throughout the main purpose is to present a fair and
impartial discussion of the important questions of the day, and
to give a large amount of useful, practical information, rather
than to devote extended space to abstract theory.

Among the important features of the new edition are a
discussion of the law of satiety, final utility, and its relation-
ship to expenses of production in the theory of value; an
explanation of the industrial system wherein the time element
has created a different organization from that of primitive
society; an adjustment of consumption to the general eco-


AMERICAN BOOK COMPANY

(189)
CHEMISTRIES
By F. W. CLARKE, Chief Chemist of the United States Geological Survey, and L. M. DENNIS, Professor of Inorganic and Analytical Chemistry, Cornell University

Elementary Chemistry | Laboratory Manual

THESE two books are designed to form a course in chemistry which is sufficient for the needs of secondary schools. The TEXT-BOOK is divided into two parts, devoted respectively to inorganic and organic chemistry. Diagrams and figures are scattered at intervals throughout the text in illustration and explanation of some particular experiment or principle. The appendix contains tables of metric measures with English equivalents.

Theory and practice, thought and application, are logically kept together, and each generalization is made to follow the evidence upon which it rests. The application of the science to human affairs, its utility in modern life, is also given its proper place. A reasonable number of experiments are included for the use of teachers by whom an organized laboratory is unobtainable. Nearly all of these experiments are of the simplest character, and can be performed with home-made apparatus.

The LABORATORY MANUAL contains 127 experiments, among which are a few of a quantitative character. Full consideration has been given to the entrance requirements of the various colleges. The left hand pages contain the experiments, while the right hand pages are left blank, to include the notes taken by the student in his work. In order to aid and stimulate the development of the pupil's powers of observation, questions have been introduced under each experiment. The directions for making and handling the apparatus, and for performing the experiments, are simple and clear, and are illustrated by diagrams accurately drawn to scale.

AMERICAN BOOK COMPANY
<table>
<thead>
<tr>
<th>Date Due</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>MAY 14 1965</td>
<td></td>
</tr>
<tr>
<td>QTR LOAN</td>
<td></td>
</tr>
<tr>
<td>SEP 16 78</td>
<td></td>
</tr>
<tr>
<td>QTR LOAN</td>
<td></td>
</tr>
<tr>
<td>DEC 8 78</td>
<td></td>
</tr>
<tr>
<td>SEC LIB JAN 29 79</td>
<td></td>
</tr>
</tbody>
</table>

PRINTED IN U. S. A.